Preference relations on actions and criteria in multicriteria decision making

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Let A be a finite set of feasible actions which are judged following several criteria. An outranking relation is defined on A by considering preference of the decision maker as a weak order on each criterion and the relation among criteria as a semi-order on the given set of criteria.

Several ways of constructing outranking relations have been proposed. One of the most popular, introduced by B. Roy, for instance ELECTRE(s), is based on the use of weights related to criteria. In our approach, the knowledge of weights is replaced by the existence of a semi-order.

A case study is developed. It deals with a computer selection problem.

1. The binary data

Let A be a finite set of feasible actions labelled $\{a, b, \dots\}, \#A = m$. A may be considered as a set of objects, alternatives, decisions, events, candidates,... .

Let us suppose that the consequences of the actions can be analysed through a set of criteria. For each criterion k, $k \in \{1, \dots, K\}$, we consider a preference relation on the set A, defined as a *weak order* S_k: the transitive relation $S_k = (I_k \cup P_k)$ is complete; P_k is antisymmetric, I_k is symmetric. The digraph corresponding to the weak order has:

(i) one arc from vertex a to vertex b iff aP_kb,

(ii) one arc from a to b and from b to a iff aI_kb.

The weak order S_k corresponds to a k-ranking of the actions, if ties are allowed.

In the classical approach of multiple objective decision making, the decision maker's preference among criteria are considered through a weight p_k on each criterion k.

We propose a model which deals with a preference relation on the set of criteria C, defined as a *semi-order* S [5,8]; the relation $S = (I \cup P)$ is complete, P is antisymmetric and aPb and $cPd \to aPd$ or cPd, aPb and $bPc \to aPd$ or dPc, $\forall a, b, c, d \in C$, I is symmetric.

It has been shown by Scott and Suppes [14] that there exist a real-valued function v on A such as:

$$aPb \quad \text{iff } v(a) > v(b) + \delta, \quad \delta > 0, \quad \forall a, b,$$
$$aIb \quad \text{iff } |v(a) - v(b)| \leqslant \delta.$$

In fact, there is a weak order which can be associated to v [6].

It is always possible to find a weak order W at minimum distance from the semi-order S.

If $\{W_{ij}\}$ and $\{S_{ij}\}$, $i \neq j \in C$ are the '0–1 opinion tableau'

$$S_{ij}, W_{ij} = 1, \qquad S_{ji}, W_{ji} = 0 \quad \text{iff } iPj,$$
$$S_{ij}, W_{ij} = 1, \qquad S_{ji}, W_{ji} = 1 \quad \text{iff } iIj,$$

the symmetric difference distance introduced by Kemeny [7] is defined as

$$d(S, W) = \sum_{i \neq j} |S_{ij} - W_{ij}|.$$

It can be easily shown that

$$\min_W d(S, W) \Leftrightarrow \min_W \sum_{i \neq j} \bar{S}_{ij} W_{ij},$$

where $\bar{S}_{ij} = 1$ iff $S_{ij} = 1$, $\bar{S}_{ij} = -1$ iff $S_{ij} = 0$. The linear objective $\Sigma_{i \neq j} S_{ij} W_{ij}$ (total number of disagreements) must satisfy the following constraints:

$$W_{ij} \in \{0, 1\}, \quad \forall\, i \neq j \in A,$$

$$W_{ij} + W_{ji} \geqslant 1, \quad \forall\, i \neq j \text{ (incomparability is not allowed)},$$

$$0 \leqslant W_{ij} + W_{jk} - W_{ik} \leqslant 1, \quad i \neq j,\ j \neq k,\ i \neq k \quad \text{(transitivity conditions)}.$$

This programme is solved by Michaud and Marcotorchino [9]. We now have a ranking for the set of criteria where r_k is the mean rank [2] for criterion k.

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2. The objective

Starting with K preference relations (weak orders) defined on A, the set of actions, we are looking for a *global outranking relation* O on A which reflects the judgement on the actions for each criterion and the preference relation (weak order) among criteria.

As an example, let us consider a computer selection problem (data from Fichefet [4]).

Tables 1 and 2 list the selected attributes together with the ranks (instead of the rates quoted by Fichefet) assigned by a Scientific Committee to the five computer systems (named a to e) which had not been eliminated yet at the moment of the very final decision. The preference relation among attributes is a weak order if the ranking of the attributes is substituted to their weight.

3. A new ordinal aggregation method: ORESTE

Starting from K weak orders related to the criteria k, $k \in \{1,\dots,K\}$, each action a is given a mean rank for each criterion k, $r_k(a)$.

The weak order W at minimum distance from the semi-order S (preference relation on the set of criteria) leads to the ranking $\{r_k\}$, where r_k is the mean rank for the criterion k.

Given $\{r_k(a), r_k\}$, we want to build an outranking relation $O = (I_A, P_A, R_A)$ such as:

$iP_A j$ if i is globally preferred to j ($O_{ij} = 1$,

Table 1

Attributes (k)	Weight (see [4])	Mean rank (r_k)
Global performances		
1. Capability to face batch workload	3	7.5
2. Response time in interactive mode	3	7.5
3. Capability to fulfil future plans	3	7.5
4. Extensibility	3	7.5
5. Reliability	3	7.5
6. Data communications	3	7.5
7. Continous operation without operator's intervention	2	19.5
Hardware performances		
8. Constraints related to storage	3	7.5
9. Disk Units	3	7.5
10. Peripherals	2	19.5
11. Maintenance support provided by manufacturer	3	7.5
Software performances		
12. Ease of use	3	7.5
13. Programming languages available	3	7.5
14. Quality of essential programming languages	3	7.5
15. Packages available	3	7.5
16. Quality of available documentation	3	7.5
17. Staff support provided by manufacturer	2	19.5
Installation and conversion considerations		
18. Delivery delay times	1	25.5
19. Conversion considerations	2	19.5
20. Facilities offered by vendor	2	19.5
21. Training courses offered	2	19.5
Management considerations		
22. Ease of operation	2	19.5
23. Accounting aids	2	19.5
Miscellaneous		
24. Manufacturer's fame	2	19.5
25. User rating	2	19.5
26. What is possible for scientific collaboration with manufacturer	1	25.5

Table 2
Ranking of computers for each attribute ($r_k(a)$)

Attributes	Computers				
	a	b	c	d	e
1	1	3	3	5	3
2	3	3	3	1	5
3	2.5	1	4	5	2.5
4	2	5	2	2	4
5	3	3	3	3	3
6	3	1	3	3	5
7	3	3	3	3	3
8	3	1	3	3	5
9	2.5	2.5	2.5	5	2.5
10	1	5	3	3	3
11	3	3	3	3	3
12	5	1	4	3	2
13	4.5	4.5	2	1	3
14	5	2.5	4	1	2.5
15	3	4.5	1.5	4.5	1.5
16	2	1	4	5	3
17	4	5	2.5	2.5	1
18	3	3	3	3	3
19	3.5	3.5	3.5	1	3.5
20	3	5	3	3	1
21	3	3	3	3	3
22	2	1	3.5	5	3.5
23	1	5	3	3	3
24	3.5	3.5	5	1.5	1.5
25	3	1	4.5	4.5	2
26	4.5	3	1.5	4.5	1.5

$O_{ji} = 0$),
$iI_A j$ if i is globally indifferent to j
($O_{ij} = O_{ji} = 1$);
$IR_A j$ if i is globally incomparable to j
($O_{ij} = O_{ji} = 0$).

The decision aid model ORESTE [10] proceeds in three steps:

Step 1 (Projection): Considering an arbitrary origin 0, a distance $d(0, a_k)$ is defined with the use of $(r_k(a), r_k)$ such that

$$d(0, a_k) < d(0, b_k) \quad \text{if } aP_k b.$$

Step 2 (Ranking): Using $d(0, a_k)$ the set of pairs (a, k) are ranked:

$$1 \leqslant R(a_k) \leqslant mK.$$

$R(a_k)$ is the mean rank for (a, k) such as

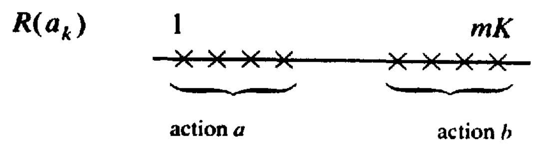
$$R(a_k) \leqslant R(b_l) \quad \text{if } d(0, a_k) \leqslant d(0, b_l).$$

Step 3 (Aggregation): If

$$C(a, b) = \sum_{k: aP_k b} [R(b_k) - R(a_k)],$$

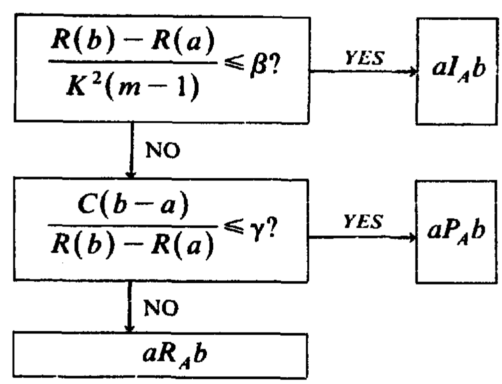
$$R(a) = \sum_k R(a_k)$$

it is easily shown that $C(a, b) - C(b, a) = R(b) - R(a)$. Moreover, $R(b) - R(a)$ is maximum in the following situation:



and, in that case, $R(b) - R(a) = K^2(m-1)$.

The aggregation procedure works as follows: Calculate $R(a)$, $C(b, a)$ and obtain the outranking relation $S_A = (I_A, P_A, R_A)$ in such a way that:
If $R(a) \leqslant R(b) \rightarrow aI_A b$ or $aP_A b$ or $aR_A b$:



where β stands for an indifference level and γ for an incomparibility level.

When $\gamma = \infty$, the outranking relation is a semi-order which becomes a weak order if $\beta = 0$.

4. Choice of the distance $d(0, a_k)$

The distance $d(0, a_k)$ must satisfy the following condition:

$$d(0, a_k) < d(0, b_k) \quad \text{if } aP_k b.$$

The 'city-block' distance is adequate:

$$d_\alpha(0, a_k) = \alpha r_k(a) + (1 - \alpha) r_k$$

where α stands for a 'substitution rate'.

If the ranges of r_k, $1 \leqslant r_k \leqslant K$, and $r_k(a)$, $1 \leqslant r_k(a) \leqslant m$, are different ($K \gg m$ or $K \ll m$) the modified city-block distance d' may be considered:

$$d'(0, a_k) = \alpha K r_k(a) + (1 - \alpha) m r_k.$$

Another alternative is the use of the Hölder metric

$$d_R(0, a_k) = \left\{\tfrac{1}{2}[r_k(a)]^R + \tfrac{1}{2}[r_k]^R\right\}^{1/R}$$

where the parameter R can be selected according some conjunction degree between actions and criteria as defined by Dujmović [3]. This type of distance was used by Van Velthoven [15] in relation with ORESTE.

5. ORESTE and Arrow's impossibility theorem

Arrow [1] listed conditions which a 'reasonable' aggregation procedure should satisfy. One of these axioms is known as the 'Pareto principle' or 'citizen's sovereignty': whenever action a ranks over action b for each criterion, then action a ranks globally over action b.

If $\beta < 1/K(m-1)$, the 'Pareto principle' holds.

This can be easily proved:

$$\begin{aligned} aP_kb, \ \forall k \ &\to d(0, a_k) < d(0, b_k), \ \forall k \\ &\to R(a_k) < R(b_k), \ \forall k \\ &\to R(b) \geqslant R(a) + K \\ &\to \frac{R(b) - R(a)}{K^2(m-1)} \geqslant \frac{1}{K(m-1)} > \beta \\ &\to aP_Ab. \end{aligned}$$

The 'stability axiom' or 'independance of irrelevant alternatives' is generally violated but practice shows that this axiom is not necessarily realistic (see [13]).

6. ORESTE and decision aid

Multi-criteria decision aid relying on the outranking relation concept has been extensively studied by Roy (see [12,13]).

Table 3

	a		b		c		d		e	
1	1.332	5	1.832	38.5	1.832	38.5	2.332	65.5	1.832	38.5
2	1.832	38.5	1.832	38.5	1.832	38.5	1.332	5	2.332	65.5
3	1.707	21.5	1.332	5	2.082	54	2.332	65.5	1.707	21.5
4	1.582	14.5	2.332	65.5	1.582	14.5	1.582	14.5	2.082	54
5	1.832	38.5	1.832	38.5	1.832	38.5	1.832	38.5	1.832	38.5
6	1.832	38.5	1.332	5	1.832	38.5	1.832	38.5	2.332	65.5
7	3.563	93.5	3.563	93.5	3.563	93.5	3.563	93.5	3.563	93.5
8	1.832	38.5	1.332	5	1.832	38.5	1.832	38.5	2.332	65.5
9	1.707	21.5	1.707	21.5	1.707	21.5	2.332	65.5	1.707	21.5
10	3.063	74	4.063	119.5	3.563	93.5	3.563	93.5	3.563	93.5
11	1.832	38.5	1.832	38.5	1.832	38.5	1.832	38.5	1.832	38.5
12	2.332	65.5	1.332	5	2.082	54	1.832	38.5	1.582	14.5
13	2.207	58.5	2.207	58.5	1.582	14.5	1.332	5	1.832	38.5
14	2.332	65.5	1.707	21.5	2.082	54	1.332	5	1.707	21.5
15	1.832	38.5	2.207	58.5	1.457	10.5	2.207	58.5	1.457	10.5
16	1.582	14.5	1.332	5	2.082	54	2.332	65.5	1.832	38.5
17	3.813	112	4.063	119.5	3.438	82.5	3.438	82.5	3.063	74
18	4.428	125.5	4.428	125.5	4.428	125.5	4.428	125.5	4.428	125.5
19	3.688	107.5	3.688	107.5	3.688	107.5	3.063	74	3.688	107.5
20	3.563	93.5	4.063	119.5	3.563	93.5	3.563	93.5	3.063	74
21	3.563	93.5	3.563	93.5	3.563	93.5	3.563	93.5	3.563	93.5
22	3.313	80.5	3.063	74	3.688	107.5	4.063	119.5	3.688	107.5
23	3.063	74	4.063	119.5	3.563	93.5	3.563	93.5	3.563	93.5
24	3.688	107.5	3.688	107.5	4.063	119.5	3.188	78.5	3.188	78.5
25	3.563	93.5	3.063	74	3.938	113.5	3.938	113.5	3.313	80.5
26	4.803	129.5	4.428	125.5	4.053	115.5	4.803	129.5	4.053	115.5

Methods to solve the following problems have been proposed:

(i) separate 'good' actions from 'bad' actions. This can be obtained by determining the kernel or the quasi-kernel of the graph induced by the outranking relation [11]

(ii) cluster actions of A in an ordered sequence of indifference classes ranging from 'best' to 'worst'. This can be obtained with the use of direct and indirect ranking [13] or be searching a weak order at minimum distance from the outranking relation as indicated in Section 1.

7. ORESTE and the computer selection problem

Let us reconsider the problem of Section 2. If distance d'_K is used with $\alpha = 0.25$; one can obtain $d'_\alpha(0, a_k)$ and $R(a_k)$ from Table 3. $R(\mathrm{a}) = 1682$, $R(\mathrm{b}) = 1683.5$, $R(\mathrm{c}) = 1747$, $R(\mathrm{d}) = 1733$, $R(\mathrm{e}) = 1669.5$, $1/K(m-1) = 0.008$. Let us suppose that $\gamma = \infty$. We obtain the following outranking relations (weak orders) respectively for $\beta < 0.0024$ and $0.0024 \leqslant \beta < 0.008$:

$$\mathrm{e} > \mathrm{a} > \mathrm{b} > \mathrm{d} > \mathrm{c}$$

and

$$\mathrm{e} > (\mathrm{a} \sim \mathrm{b}) > \mathrm{d} > \mathrm{c}.$$

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