Vagueness

An exercise in logical analysis

BY

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"Vagueness and accuracy are important notions, which it is very necessary to understand." (Bertrand Russell).

"The notation, however, is what we lack, and the verdict of the mere feeling is liable to fluctuate." (Henry James).

I. INTRODUCTION

IT IS a paradox, whose importance familiarity fails to diminish, that the most highly developed and useful scientific theories are ostensibly expressed in terms of objects never encountered in experience. The line traced by a draughtsman, no matter how accurate, is seen beneath the microscope as a kind of corrugated trench, far removed from the ideal line of pure geometry. And the "point-planet" of astronomy, the "perfect gas" of thermodynamics, or the "pure species" of genetics are equally remote from exact realization. Indeed the unintelligibility at the atomic or sub-atomic level of the notion of a rigidly demarcated boundary shows that such objects not merely are not but could not be encountered. While the mathematician constructs a theory in terms of "perfect" objects, the experimental scientist observes objects of which the properties demanded by theory are and can, in the very nature of measurement, be only approximately true. As Duhem remarks, mathematical deduction is not useful to the physicist if interpreted rigorously. It is necessary to know that its validity is unaltered when the premise and conclusion are only "approximately true."[1] But the indeterminacy thus introduced, it is

[1] P. Duhem: "... une deduction mathématique n'est pas utile au physicien tant qu'elle se borne à affirmer que telle proposition, rigoureusement vraie, a pour conséquence l'exactitude de telle autre proposition. Pour être utile au physicien, il lui faut encore prouver que la seconde proposition rest *à peu près* exacte lorsque la première est seulement *à peu près* vraie" (*La Théorie Physique*, p. 231).

necessary to add in criticism, will invalidate the deduction unless the permissible limits of variation are specified. To do so, however, replaces the original mathematical deduction by a more complicated mathematical theory in respect of whose interpretation the same problem arises, and whose exact nature is in any case unknown.

This lack of exact correlation between a scientific theory and its empirical interpretation can be blamed either upon the world or upon the theory. We can regard the shape of an orange or a tennis ball as imperfect copies of an ideal form of which perfect knowledge is to be had in pure geometry or we can regard the geometry of spheres as a simplified and imperfect version of the spatial relations between the members of a certain class of physical objects.[2] On either view there remains a gap between scientific theory and its application, which ought to be, but is not, bridged. To say that all language (symbolism, or thought) is vague is a favorite method for evading the problems in-involved and lack of analysis has the disadvantage of tempting even the most eminent thinkers into the appearance of absurdity. Duhem claims that "for the strict logician," a physical law is neither true nor false.[3] For Einstein mathematics is either uncertain or inapplicable,[4] and Russell cheerfully sacrifices logic as well.[5]

The aim of this paper is to avoid such wholesale destruction of the formal sciences by supplying in greater detail than has hitherto been attempted an analysis and symbolism for the "vagueness" or "lack of precision" of a language.

[2] Plato: "Those who study geometry and calculation . . . use the visible squares and figures, and make their arguments about them, though they are not thinking about them, but about those things of which the visible are images. Their arguments concern the real square and a real diagonal, not the diagonal which they draw, and so with everything. The actual things which they model and draw . . . they now use as images in their turn, seeking to see those very realities which cannot be seen except by the understanding." (*Republic*, 510—Lindsay's translation.)

[3] "Toute loi physique est une loi approchée; par conséquent, pour le strict logician, elle ne peut être, ni vraie, ni fausse." (Loc. cit. p. 280.)

[4] "Insofern sich die Sätze der Mathematik auf die Wirklichkeit beziehen sind sie nicht sicher, und insofern sie sicher sind, beziehen Sie sich nicht auf die Wirklichkeit." (*Geometrie und Erfahrung, p. 3.*)

[5] "All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life, but only to an imagined celestial existence." (*Vagueness, Australasian Journal of Philosophy*, Vol. 1 (1923), p. 88.)

In this paper Russell contends that "all language is more or less vague." Again the "laws of Excluded Middle is true when precise symbols are employed but it is not true when symbols are vague, as, in fact, all symbols are." (Ibid. p. 85.)

We shall not assume that "laws" of logic or mathematics prescribe modes of existence to which intelligible discourse must necessarily conform. It will be argued, on the contrary, that deviations from the logical or mathematical standards of precision are all pervasive in symbolism; that to label them as subjective aberrations sets an impassable gulf between formal laws and experience and leaves the *usefulness* of the formal sciences an insoluble mystery. And it is the purpose of the constructive part of the paper to indicate in outline an appropriate symbolism for vagueness by means of which deviations from a standard can be absorbed by a re-interpretation of the same standards in such a way that the laws of logic in their usual absolutistic interpretation appear as a point of departure for more elaborate laws of which they now appear as special or limiting cases. The method yields a process by which deviations, when recognized as such, can be absorbed into the formal system. At every stage the mathematics we already employ will provide the material for the increasing accuracy of the next stage.

It is one of the paper's main contentions that with the provision of an adequate symbolism the need is removed for regarding vagueness as a defect of language. The ideal standard of precision which those have in mind who use vagueness as a term of reproach, when it is not a shifting standard of a relatively less vague symbol, is the standard of scientific precision. But the indeterminacy which is characteristic of vagueness is present also in all scientific measurement. "There is no experimental method of assigning numerals in a manner which is free from error. If we limit ourselves strictly to experimental facts we recognize that there is no such thing as true measurement, and therefore no such thing as an error involved in a departure from it."[6] Vagueness is a feature of scientific as of other discourse.

The impressionist painting of a London street in a fog is not a vague representation of what the artist sees, since his skill largely consists in the accuracy with which the visual impression is transcribed. But the picture is called vague in relation to a hypothetical laboratory record of the wave lengths and positions of the various objects in the street, while it is forgotten that that record, in supplying additional detail, obliterates just those large scale relations in which the artist or another observer may be interested. This paper is written to show that while the vague symbol has a part to play in language which cannot be equally well performed by more accurate symbols from another level (wave

[6] N. R. Campbell, *Measurement and Calculation*, p. 131.

lengths as a substitute for names of colors) the transition to levels of higher accuracy can always in principle be made.

2. SUMMARY OF THE PAPER'S ARGUMENT

The process of logical analysis of a language can be regarded as the exhibition of a set of conventions for the use of symbols, abstracted from the regularity of linguistic habits in some postulated speech community, and proceeding by a series of successive approximations involving the use of "simplified" or "model" entities.

The vagueness of symbols in any such abstract system is a symptom of the degree of deviation of the "model" language from the empirically discoverable linguistic habits in the corresponding speech community.

A typical example of vagueness is described. A symbol's vagueness is held to consist in the existence of objects concerning which it is intrinsically impossible to say either that the symbol in question does, or does not, apply. The set of all objects about which a decision as to the symbol's application is intrinsically impossible is defined as the "fringe" of the symbol's field of application. It is claimed that all symbols whose application involves the recognition of sensible qualities are vague, and a typical case is constructed for convenience of reference. Vagueness is distinguished from generality and from ambiguity. The former is constituted by the application of a symbol to a multiplicity of objects in the field of reference, the latter by the association of a finite number of alternative meanings having the same phonetic form; but it is characteristic of the vague symbol that there are no alternative symbols in the language, and its vagueness is a feature of the boundary of its extension, and is not constituted by the extension itself. Russell's definition of vagueness (in a paper to which frequent reference is made) as constituted by a one-many relation between symbolizing and symbolized systems is held to confuse vagueness with generality.

The assumption of the existence of a well-defined set of objects to which the application of a vague symbol is doubtful is shown to be inconsistent with the usual meaning of negation, and the conclusion is shown to follow whether the number of individuals in the field of reference is finite or infinite. But it is shown that there is no good reason to regard the defining characteristic of vagueness as "subjective." The crude notion of the "fringe" is therefore replaced by a statistical analysis of the frequency of deviations from strict uniformity by the "users" of a vague symbol. In this, the most important section of the paper it is found possible to define the notion of a consistency profile or, its equivalent, a consistency function, corresponding to each vague symbol and thus to classify, or even, theoretically, to measure, degrees of vagueness.

The definition of consistency profile is based on the existence of "a group of users of a language" whose linguistic habits are sufficiently stable and inter-correlated to permit of limiting assertions concerning frequencies of deviations from a standard. The definition of the users of a symbol is shown to be another aspect of the definition of the symbol itself, and the relation between the two processes is illustrated by analogy with the definition of a biological species. An experiment is described whose results illustrate the construction of a consistency profile, and the analysis is extended to the consideration of logical relations between vague symbols.

3. VAGUENESS DESCRIBED

The vagueness of a term is shown by producing "borderline cases," i.e., individuals to which it seems impossible either to apply or not to apply the term. Thus a word's vagueness is usually indicated, more or less explicitly, by some statement that situations are conceivable in which its application is "doubtful" or "ill-defined," in which "nobody would know how to use it" or in which it is "impossible" either to assert or deny its application.

Peirce's definition[7] is admirably clear: "a proposition[8] is vague when there are possible states of things concerning which it is *intrinsically uncertain* whether, had they been contemplated by the speaker, he would have regarded them as excluded or allowed by the proposition. By intrinsically uncertain we mean not uncertain in consequence of any ignorance of the interpreter, but because the speaker's habits of language were indeterminate."[9] An example will now be discussed in more detail.

[7] Baldwin's *Dictionary of Philosophy and Psychology*, II, 748.

[8] In this paper reference will always be made to the vagueness of a word or symbol, but no important difference is involved in speaking of a proposition's vagueness. The proposition can be regarded as a complex symbol and its vagueness defined in terms of that of its constituents, or vice versa.

[9] In the remainder of the passage Peirce explains that by an indeterminacy of habits he means the hypothetical variation by the speaker in the application of the proposition, "so that one day he would regard the proposition as excluding, another as admitting, those states of things." But the knowledge of such variation could only be "*deduced* from a perfect knowledge of his state of mind; for it is precisely because these questions never did, or did not frequently, present themselves, that his habit remained indeterminate."

Let us consider the word *chair*, say. On reflection, one is impressed by the extraordinary variety of objects to which the same name is applied: "... think of arm chairs and reading chairs and dining-room chairs, and kitchen chairs, chairs that pass into benches, chairs that cross the boundary and become settees, dentist's chairs, thrones, opera stalls, seats of all sorts, those miraculous fungoid growths that cumber the floor of the arts and crafts exhibitions, and you will perceive what a lax bundle in fact is this simple straightforward term. In co-operation with an intelligent joiner I would undertake to defeat any definition of chair or chairishness that you gave me."[10]

It is important in such a case that the variety of application to objects differing in size, shape and material should not be confused with the vagueness of the word. The variety of application no doubt arises from the fact that chairs are defined by the need to be satisfied. "... Every common noun, every concept is essentially merely an affective grouping. In a plurality of objects, differing from the point of view of perception even very widely from one another, we discover the same capacity to satisfy some given affectivity, some given need or desire of ours, and through this capacity we reduce this very plurality to a unity."[11] Being "a separate seat for one," as the dictionary puts it, is compatible with much variation in form and material.

But in speaking of the vagueness of the word chair, attention is directed only to the fact that objects can be presented whose membership of the class of chairs is incurably "uncertain" or "doubtful." It is the indeterminacy of the usage, not its extension, which is important for the purpose of the argument. The finite area of the field of application of the word is a sign of its *generality*, while its vagueness is indicated by the finite area and lack of specification of its boundary.[12] It is because *small* variations in character are unimportant to success in serving

[10] H. G. Wells, *First and Last Things*, p. 16.

[11] E. Rignano, *Psychology of Reasoning*, p. 109.

[12] Cf. B. A. W. Russell "A vague word is not to be identified with a general word" (*Analysis of Mind*, p. 184). He adds, however, "that in practice the distinction is apt to be blurred" and blurs it himself in saying "a memory is vague when it is appropriate to *many* occurrences" (*Loc. cit.* p. 182). This confusion between generality and vagueness invalidates his neat definition "the fact that meaning is a one-many relation is the precise statement of the fact that all language is more or less vague." (*Vagueness*, p. 89.)

The confusion may ultimately be traced to a certain uneasy nominalism in Russell's philosophy which tends to treat generality and vagueness indifferently as imperfections of symbolism in relation to the attempt to describe a universe composed exclusively of absolutely specific or atomic facts.

the purpose of being "a separate seat for one" that it is possible, by successive small variations in any respect, ultimately to produce "borderline cases." The cumulative action of such variation in producing large additive effects is at the root of the felt inability either to withhold or to apply a general term to the unusual and the extreme case.

One can imagine an exhibition in some unlikely museum of applied logic of a series of "chairs" differing in quality by least noticeable amounts.[13] At one end of a long line, containing perhaps thousands of exhibits, might be a Chippendale chair: at the other, a small nondescript lump of wood. Any "normal"[14] observer inspecting the series finds extreme difficulty in "drawing the line" between chair and not-chair. Indeed the demand to perform this operation is felt to be inappropriate *in principle*: "*chair* is not the kind of word which admits of this sharp distinction" is the kind of reply which is made "and if it were, if we were forbidden to use it for any object which varied in the slightest way from the limiting term, it would not be as useful to us as it is." This is the sensible attitude but it raises difficulties for logic.

In order to circumvent these difficulties, we shall make use of the fact that the uncertainty of a single normal observer, or the variation in the decisions made by a number of such observers, either of which can be taken as the definition of vagueness, is a matter of degree, varying quantitatively, though not regularly, with the position of an object in the series. At the extremities of the series little or no uncertainty is felt, but the observer grows increasingly doubtful when the borderline cases in the center are approached; "everybody" agrees that the Chippendale chair *is* a chair, "nobody" wants to sit upon, still less to call a chair, a shapeless lump of wood, but in intermediate cases personal uncertainty is a reflection of objective lack of agreement.

We have used alternative but correlated definitions of vagueness in order not to prejudge the issue whether vagueness is subjective or objective.[15] On the one hand we can use an observer's feelings or report of his feelings; on the other, the set of divisions made by a set of inde-

[13] The variation of this amount with the choice of the observer, and with conditions affecting the same observer, strengthens the subsequent argument by introducing further indeterminacy into the operation of "drawing the line."

[14] This is, in part, a definition of the "normal" observer; we shall reject the testimony of an observer who claimed to have discovered *the* point at which the division was to be made. Cf. section 7 for a fuller discussion of this point.

[15] See section 5, below.

pendent observers who are given sufficient inducement to make a unique division in the series irrespective of their feelings of uncertainty.[16]

The vagueness of the word *chair* is typical of all terms whose application involves the use of the senses. In all such cases "borderline cases" or "doubtful objects" are easily found to which we are unable to say either that the class name does or does not apply. The case of a color name, whose relative simplicity is unobscured by the variation in application of such "artificial" names as chairs, is specially striking. If a series of colored cards of uniform saturation and intensity are arranged according to shades ranging by least perceptible differences from reds through oranges to yellows, the "uncertainty" which is typical of vagueness is at once demonstrated. "The changes of color in the spectrum are throughout so continuous that *it is not possible to find the exact point at which the changes of direction begin*."[17] It would be easy, but uninstructive, to multiply examples. Reserving the terms of logic and mathematics for separate consideration,[18] we can say that all "material" terms, all whose application requires the recognition of the presence of sensible qualities, are vague in the sense described.

4. LOCATION OF THE FRINGE

The quantitative variation in the degree of uncertainty felt by a typical observer, or the equivalent variation in the divisions made by a number of observers, will be used later as the basis of a method for symbolizing vagueness. But before doing this it is necessary to dispose of a plausible but mistaken view which seeks to solve the problem of border line cases by allocating them to a region of "doubtful application," a kind of no man's land lying between the regions when a term applies and does not apply. For even if it is granted that all material terms are vague in the sense described it might still be said that the existence of border line cases is unimportant. Such cases occur so infrequently, it might be argued, that consideration can always be restricted to objects concerning which the "doubt" does not arise. To such objects difficulties concerning the indeterminacy of the boundary will

[16] Cf. the experiment described in Appendix I when the subject agrees beforehand to make a unique division, the "inducement" being desire to keep his word, or curiosity, or some other motive.

[17] Stout, *Manual of Psychology*, p. 160. This manner of phrasing the situation suggest of course that the fault is in the language or in imperfect perception: there *is* an "exact point" where the transition occurs but we are unable to find it.

[18] See Appendix II.

not be relevant; for these cases will remain unproblematic whichever separation is made in the field of application, and since we do not choose to argue about borderline cases no difficulty remains.

An objection of this sort misses the point: we do not claim to have discovered a serious practical difficulty, but are trying to achieve the accomodation of an unduly simple conception of logic to the undoubted practical efficacy of formally invalid classificatory procedure. The presupposition of the existence of a class of "doubtful" objects will involve the assumption either of an exact boundary or of a doubtful region (of the second order)[19] between the fringe and the class of unproblematic objects.

Either assumption will be shown to be incompatible with the usual definition of negation, and thus indirectly incompatible with the strict application of logical principles. The exposition will be simplified by using a set of constant symbols to illustrate the features of vagueness described in the last section. Let *L* then, be a typical example of a vague symbol. It has been seen that the vagueness of *L* consists in the impossibility of applying *L* to certain numbers of a series. Let the series *S*, say, be linear, and composed of a finite number,[20] ten say, of terms *x*, let the rank of each term in the series be used as its name (so that the constant values of the variable *x* are the integers one to ten inclusive). Finally let the region of "doubtful application" or "fringe" be supposed to consist of the terms whose numbers are five and six respectively. There is, of course, no special significance in the choice of the numbers, ten, five, six, which are taken simply for the convenience of having definite numbers to which to refer.

In the usual notation of the propositional calculus Lx will mean *L applies to X* and ∼Lx will mean *L does not apply to x* or *Lx is false.* (synonymous expressions)

Suppose now that L1, L2, L3, L4 are true, while L5 and L6 are "doubtful." It can only follow that to assert Lx of any x is positively to exclude it only from the range 7 to 10, since we cannot be sure, when Lx is asserted, that x does not perhaps occur in the range 5, 6. Thus to assert Lx is tantamount to confining x to the range 1 to 6.

Having obtained this result, it is easy to construct a similar argument in respect of ∼Lx. The assertion of ∼Lx can, no more than the assertion of Lx, positively exclude x from the fringe 5, 6. It follows that to

[19] Russell assumes an infinite series of doubtful regions, each fringe having a fringe of higher order at its boundary, but does not pursue the consequences of this assumption.

[20] The hypothesis of an infinite series is considered later in this section.

assert $\sim$Lx is tantamount to excluding x from the range 1 to 4 and confining it to the range 5 to 10.

In short, inability to find a logical interpretation of doubtful and perhaps in terms of the two truth values, truth and falsehood, forces us to admit that the ranges of application of Lx, 1 to 6, and of $\sim$Lx, 5 to 10, overlap in the fringe, 5, 6.

On the other hand, the statement $\sim$Lx is, by definition of the logical operation of negation, true only when Lx is false, and false only when Lx is true. If, as we have assumed, asserting Lx confines x to the range 1 to 6, Lx is false only when x belongs to the range 7 to 10. Thus in contradiction to our previous result that $\sim$Lx is true when and only when x belong to the range 5 to 10, $\sim$Lx should be true when and only when x belongs to the range 7 to 10. The formal properties of logical negation are incompatible with an interpretation which allows the domain and the complementary domain of a propositional funtion to overlap.

We can clinch the argument by attempting to translate the definition of L's vagueness, in some such form as *there is at least one term to which neither L nor its contradictory applies*, into the symbolism of the propositional calculus. Translating the italicized phrase in the last sentence gives

$$(\exists x)\ \{\sim L(x)\cdot\sim(\sim L(x))\}$$

which is at once transformed, by the rule of double negation, into

$$(\exists x)\ \{\sim L(x)\cdot L(x)\}$$

which is a contradiction. Such a contradiction is only to be evaded by denying the equivalence of $\sim(\sim L(x))$ and $L(x)$, i.e. by refusing to identify the operation $\sim$ when prefixed to a vague symbol with the ordinary operation of negation. This point of view will be incomplete and unplausible unless it is possible to define the new sense of $\sim$, i.e. to give the rules according to which the sign is to be used.

The situation is in some ways comparable to the re-interpretation of negation in mathematics arising from the criticisms of the Intuitionist school of philosophers of mathematics. Refusal to accept a theorem and its contradictory as exclusive alternatives forces the Intuitionists to construct a logical calculus, in which ordinary negation is eliminated, its function being taken by a new notion differing in its formal properties.[21] Whereas, however, the Intuitionists, setting out from a fairly

[21] Cf. M. Black, "The Claims of Intuitionism," *The Philosopher*, July 1936.

well defined criterion of constructibility are able to invent an appropriate calculus, our present investigation is still in the more rudimentary stage of knowing simply that "the notation is what we lack" and "the verdict of the mere feeling" is liable not merely to fluctuate but to lead to contradictions.

This part of the discussion should be completed by showing that we are bound to reach the same overlapping of domains and hence the same contradiction, if we were to allow the fringe to be itself bounded by a fringe of higher order and that in turn by another and so on ad infinitum.[22]

It will be sufficient to consider the case of a linear continuum e.g. the set of all geometrical points from a point *a* upon a straight line to a point *b* on the same straight line. If there is a series of fringes each limited by a subsidiary fringe (all composed of points between *a* and *b*) there must be two points *c* and *d*, which may be identical with *a* and *b* respectively, beyond which *no* fringe extends. If we choose *c* and *d* to be as close together as possible,[23] the assertion of Lx will assign x to the interval *a* to *d*, and the assertion of ∼Lx to the interval *c* to *b*, these ranges overlapping as in the argument for the finite case. In either case, whether the number of terms in the field of reference is finite or infinite, denial of the existence of a unique boundary between the domains of Lx and ∼Lx leads to contradiction. Thus it is impossible to accept Russell's suggestion that the fringe itself is ill-defined.[24] Ill-defined can only mean undefined—there is no place for a *tertium quid* in traditional logic. But an undefined fringe means absence of all specification of boundary between the fields of application of a term and its contradictory—and this is in flagrant contradiction with the facts of the ordinary use of language. *Red* and *yellow* are used as distinct, not identical, symbols in a way which is not seriously affected by the existence of continuous gradations between the two colors.

On the other hand, the awkwardness of assuming a well defined boundary to the fringe is shown clearly in the classical paradox of the heap, sometimes attributed to Zeno (Burnet; *Greek Philosophy*, p. 114, 5). The argument is paraphrased by Adamson (*Development of Greek Philosophy*, p. 38) in this way. "A measure of corn when thrown out makes a sound. Each grain and each smallest part of a grain must

[22] This is Russell's assumption in the paper to which reference has already been made.

[23] The argument would need trivial adjustments if the field of reference, while having an infinite number of terms, did not constitute a continuum.

[24] Op. cit., p. 88.

therefore have made a sound yet no sound is made by a single grain." What is essentially the same argument sometimes appears in modern dress as the paradox of the bald man. Plucking a single hair from a man's head cannot make him bald if he is not so before. But the plucking of all his hair will make him bald and this can be accomplished by the successive pluckings of single hairs.

Both forms of the paradox are associated with the emergence of qualities as a result of successive small alterations in respect of some other (quantitative) characteristic, none of which, except the last, produce *any* change in quality. The repugnance felt towards this type of discontinuity may be merely a prejudice but it seems to be more, and I am unaware of any satisfactory discussion of it. So long as this type of argument is held to apply to a few vague terms the matter is not serious, but if we admit *all* terms are vague its application will invalidate any deductive argument into which it is inserted, and is as awkward for logic as the notorious mathematical antinomies are for mathematics.

The difficulty is serious enough. If we are right in our claim that all material terms are vague, the formal apparatus of logic (and indirectly of mathematics, though this has not been shown) seems to break down. We are unable even to assume that Lx is incompatible with $\sim$Lx, without the assurance that x is not in the fringe, and we are unable to say when the fringe begins and ends. The attempt to assert that x does not belong to L's fringe, say A('Lx') leads us into an infinite regress

$$A\{`A(`Lx')\}, A\{`A\{`A(`Lx')\}\}, \text{etc.}$$

and does not evade the difficulty. To say, as Russell does, that "all traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestial life, but only to an imagined celestial existence"[25] is to abandon "traditional logic." If we can "imagine" precise symbols we can construct them—and if "*all* symbols are vague" we cannot even imagine precise ones.[26]

5. IS VAGUENESS SUBJECTIVE?

Subjective is here taken to mean whatever belongs to the processes of cognition, feeling or willing as distinct from whatever belongs to the notion of their object.[27] Suppose an observer O, in the presence of an

[25] Op. cit., p. 88.

[26] Cf. section 9 below when Russell's argument on this point is further discussed.

[27] Cf. article on "Objective," Baldwin's Dictionary of *Phil. and Psych.* II, 192.

environment E, utters a set of words S. We shall say that a feature of S is subjective or objective according as the fact that that feature occured in the situation which consisted of O's enunciating S is evidence for a fact about O or a fact about E respectively. Thus suppose S was uttered in an unusually loud voice; the intensity of the sounds occurring is, in the English language,[28] evidence of O's state of mind (e.g. that he is angry) but not evidence about the ostensible subject matter of his report.[29] Thus the intensity of the sounds in S is a subjective feature. Again if a sufferer from delirium tremens says *There is a pink lizard over there* we are able to deduce only that he is *seeing* a pink lizard (a fact about O), not that there *is* a pink lizard over there (a fact about E). Thus the occurrence of the terms *pink* and *lizard* in his statement are subjective features.

The distinction between subjective and objective features of an utterance is therefore closely connected with the distinction between psychological and physical data and the title of this section can be rephrased as follows: Are the defining phenomena of vagueness, i.e. the variation in the position of the boundary chosen by various observers (or the same observer at various times), facts about human behavior (psychological or sociological data) or facts about the physical world?

The question is best answered by comparison with the corresponding deviation from strict regularities of scientific instruments. For there is no essential difference in this context between the human reporter and the scientific instrument. We can regard the observer as an instrument for making division in a series of objects; the report of his feelings can then be likened to the oscillation of an instrument in a range where direct measurement is difficult.

(Conversely, the scientific instrument always needs a human observer before its readings can be incorporated into the scientific record, so that in a sense the instrument is merely a prolongation and reinforcement of the scientists' sense organs). What is needed is a description of the circumstances in which a variation of the readings supplied by an instrument reporting the character of an object E is ascribed (a) to an "error" of the instrument or (b) to a change in E. The cases (a) and (b) correspond to the variations in the reading being, in the terminology of this section, "subjective" and "objective" respectively.

[28] The qualification is necessary because some languages such as Chinese use differences of pitch as significant linguistic elements.

[29] Since O's state of mind may depend upon his environment, evidence about O's state of mind, may of course, *indirectly* yield evidence also about E.

It is clear that the use of a single instrument I measuring some physical magnitude of an object E and yielding (as all scientific instruments do) various readings on various occasions, is not sufficient for choice between cases (a) and (b) of the last paragraphs. For a character which is present in all measurement cannot serve as a criterion for discriminating between two types of measurement. The method actually used by scientists is of course to keep E and the whole of its environment except I unchanged[30] while using other instruments I′, I″ etc. in order to perform the same readings. If the variations in the readings are systematic i.e. if they conform to a law which is confirmed by all the instruments, the variation is ascribed to a change in E. If for example the length of a body is recorded at intervals of a second as alternately 1 inch and two inches by every instrument, the systematic variation would be interpreted as a periodic alteration in the body's length. In mathematical terminology, the necessary and sufficient condition for variations in the readings of an instrument to be regarded as indication of changes (objective patterns) in the object measured is that the law connecting such variations should be invariant with respect to replacement of the measuring instrument by another of a certain set of instruments. Conversely, if the variations are not connected by an invariant law, they are ascribed to changes in the instrument (subjective factors). This assumption is strengthened if a law can be discovered connecting the variations with the internal structure of the instrument for such a law allows prediction of the nature and amount of the variations irrespective of the nature of E. But in default of such a law the variations may simply be called "random," the essential being that the variations are invariant.

It is for such reasons that in physics gross deviations in measurements of position are ascribed to defects of the measuring instruments, while the joint indeterminacy of position and impulse of Quantum Mechanics, is regarded as objective.

Are the variations in the boundary decisions made by various members of a set of observers analogous to the errors of a scientific instrument or to the variations in the readings of an instrument in accordance with objective variations in the situation measured? We have assumed that the variations are not purely random and that the variant decisions exhibit some statistical regularity. If this is a justified assumption (and without it we are unable to account for the success with which vague symbols are used)—vagueness is clearly an objective feature of

[30] In practice this can never be completely successful.

the series to which the vague symbol is applied.[31] And it will be shown in the next section how the vagueness of the symbolism can be made explicit in a way which ordinary language fails to do[32] and be made in this way to serve as an adequate model of those relations in the field of application from which it arises.

6. DEFINITION OF THE CONSISTENCY—PROFILE

We propose to replace the crude and untenable distinction between fringe and region of certain application by a quantitative differentiation, admitting of degrees, and correlated with the indeterminancy in the divisions made by a group of observers.

The definition involves three fundamental notions: *language* (or *users of a language*), *a situation in which a user of a language is trying to apply a symbol L to an object x*, and *the consistency of application of L to x*. It is impossible to define them in independence of each other, for the first, which is clearly involved in the second and third, is in turn based upon the last. Thus the three notions must be defined in terms of a single process of interpretation, assigning a meaning to any context in which they are used. For the present we shall define a "language" as the vocabulary and syntax abstracted from the laws expressing the uniformity of linguistic habits of a certain group of persons; and that group of persons we call the users of the language.[33] This definition will

[31] It needs, therefore, to be clearly distinguished from such features of symbolism as ambiguity. The latter is constituted by inability to decide between a finite member of alternative meanings having the same phonetic form (homonyms). The fact that ambiguity *can be removed* shows it to be an accidental feature of the symbolism. But any attempt to remove vagueness by a translation is defeated by the over-specification of meaning thus produced. Cf. an attempt to replace *The hall was half full* by *The ratio of the number of persons in the hall to the number of seats was exactly half*. The presence of *one* person too many would falsify the second, but not the first of the statements.

[32] In ordinary language, vagueness is shown explicitly by the use of adverbs of degree- or number such as *any*, *many*, *rather*, *almost*, etc. These serve as a set of pseudo-quantifiers, generalisations as it were of the "respectable" quantifiers *all* and *any*, forming a slide ing scale which can be attached to any adjective. The method of the next section, which reduces to the conversion of propositional functions into propositional functions of an extra variable by the addition of a numerical parameter is thus the generalisation of a device already present in ordinary discourse.

[33] The "set of conventions" determining the vocabulary and syntax of such a language are the simplified expressions, in the imperative mood, of the empirically discoverable rules of usage. While the existence of such a language presupposes, by definition, *some* uniformity in the linguistic habits of its users, the empirical laws expressing the partial uniformity of such habits are complex, in process of variation, and heterogeneous in

be discussed further in the next section, when it will be shown that the whole procedure is not circular. For the present, however, language will be treated as a relatively unproblematic notion, and we proceed to explain the notion of consistency of application. The method is based on the assumption that while the vagueness of a word involves variations in its application by the users of the language in which it occurs, such variations must themselves be systematic and obey statistical laws if one symbol is to be distinguished from another. It will be necessary to refer to situations in which a user of the language makes a decision whether to apply L or $\sim$L to an object x. (Such a situation arises, for instance, when an engine driver on a foggy night is trying to decide whether the light in the signal box is really a red or a green light.) Let us call such a situation a *discrimination of x with respect to L*, or a DxL for short. (Then a DxL will be identical with a $Dx{\sim}L$, by definition.)

For some x's, the result of a DxL is almost independent of the observer; most users of the language, and the same user on most occasions, decide either that L applies or the $\sim L$ applies. In either case there is practical unanimity among competent observers as to the correct judgement. For other x's (in the "fringe") there is no such unanimity.

In any number of DxL involving the same x but not necessarily the same observer, let m be the number which issue in a judgment that L applies and n the number which issue in the judgment that $\sim$L applies. We define *the consistency of application of L to x* as the limit to which the ration m/n tends when the number of DxL and the number of observers increase indefinitely. (The second number is of course limited to the total number of the users of the language.) Since the consistency of the application, C, is clearly a function of both of L and x, it can be written in the form $C(L,x)$.

In a previous section we claimed certain systematic features in the variation of application of a vague but unambiguous symbol. It is now possible to specify these features more exactly. As we pass from left to right along the series S of terms x, the corresponding values of $C(L,x)$ will have large values at the outset (region of "certain" application of L), decrease until values near to one are reached (fringe), and

character. It is necessary to distinguish between rules of logic, grammar and good taste. The neglect of certain distinctions and discriminations habitually made by users of the language provides a simplified or "model" language bearing some, but not too much, resemblance to their actual habits. Then the first crude analysis can be corrected by a supplement which considers the facts neglected. Thus the definition proceeds by a series of successive approximations.

decrease again until values near to zero are reached (region of "certain" application of $\sim L$). A list of the exact values of $C(x,L)$ corresponding to each member x of S will be an exact description of L's vagueness. In the figure below, a typical set of consistencies is shown in graphical form.

The numbers along the horizontal axis denote the position of terms in the series S, while the height of a point above a number vertically beneath it represents the consistency of application of the symbol in question for the corresponding term of the series. The points marking the values of the consistencies associated with each member of the series

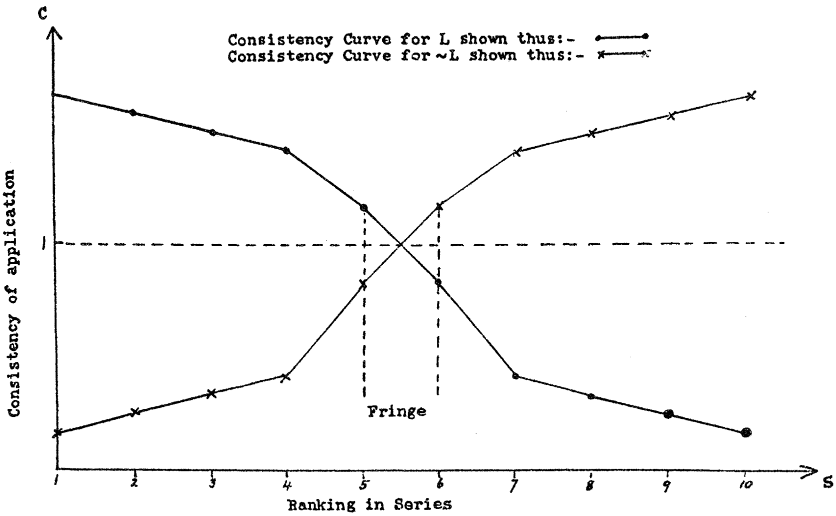

Fig. 1. Consistency of Application of a Typically Vague Symbol

have been joined to form an open polygonal line. It will be convenient to call the curve thus obtained a *consistency profile* for the application of L to the series S. In practice the number of terms in S will usually be very much greater that 10 (e.g. there are said to be something like 700 distinguishable shades of gray) and the consistency curve will approximate to a smooth curve having a continuous gradient.

The exact shape of the consistency curve will, of course, vary according to the symbol considered. It has been assumed that the typical symbol, L, is unambiguous, but an ambiguous symbol will be easily detected by the presence of more than one fringe in its consistency curve. In other words the steady decrease of consistency as we move from left to right is taken as a definition of unambiguity. Further, the introduction of consistency profile allows us to define the relative vagueness of symbols on the basis of a classification of their corresponding consistency profile. Thus the very precise symbol would have a consistency curve made up of a straight line almost parallel to the horizontal axis, and at a great distance from it, followed by a steep drop to another line almost parallel to the horizontal axis and very close to it, i.e. the curve is marked by the narrowness of the fringe and lack of variation in the symbol's application elsewhere.

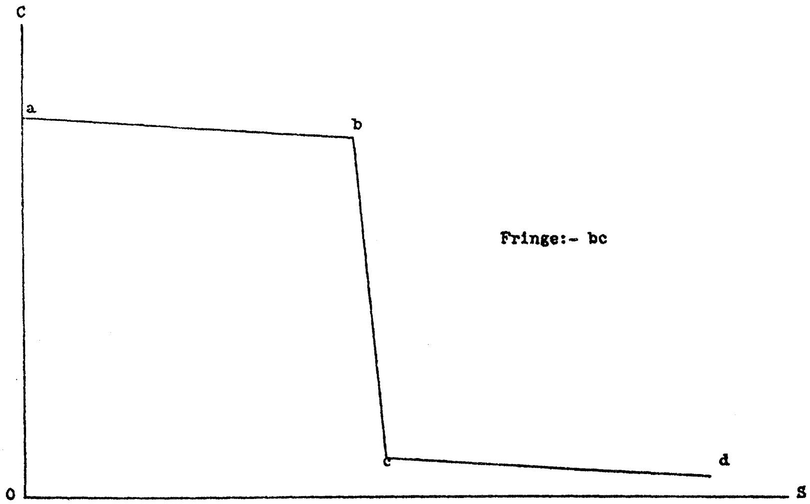

FIG. 2. CONSISTENCY CURVE OF A VERY PRECISE SYMBOL

The very vague but unambiguous, symbol, on the other hand would have a consistency profile approximating to a straight line of constant negative gradient, i.e. the fringe merges into the whole field and there is continuous variation in the symbol's application (Fig. 3).

Intermediate cases could be classified according to their deviation from the extreme types illustrated in Fig. 2 above and Fig. 3 on facing page.

In order to do this, it might be more convenient to plot not the consistency as above defined, but the deviation of the consistency from the extreme values. Let m be the number of the *more favoured* judgements in a total of n DxL, made as before with respect to the same x. And let Z be the limit approached by $2m/n$ as the number of DxL increases. Then Z lies always between 0 and 1, is high in the fringe and low in the regions of certain application, whether of L or $\sim L$. A curve connecting Z with the rank of x would approximate in shape to the frequency curves studied in statistics and the determination of a symbol's degree of vagueness would then reduce to well known statistical problems such as the determination of the area or flatness (kurtosis) of a frequency curve.

It is to be noticed that the existence of a series of relatively less vague symbols does not imply the existence of a symbol of zero vagueness[34] any more than the existence of greater lengths implies the exist-

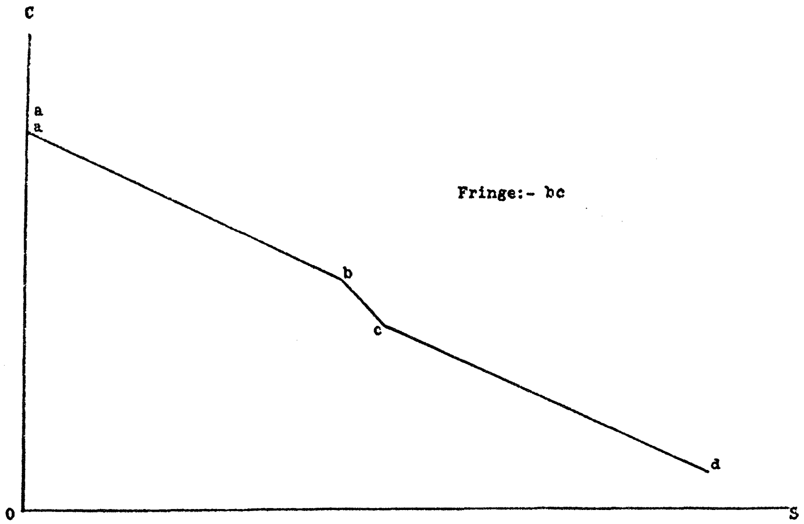

FIG. 3. CONSISTENCY CURVE OF A VERY VAGUE SYMBOL

ence of a greatest or least length.[35] The limits to the application of the term *length* are of exactly the same kind as the limits to the application of *red* or *chair* or any other vague word. It is not possible to set any upper limit to the application of the term *length*, but its applica-

[34] Cp. Russell, *Vagueness*, "we are able to conceive precision; indeed if we could not do so we could not conceive of vagueness which is merely the contrary of precision" (p. 89).

[35] The final term in any case differs from its predecessors in *some* respects. The situation is indeed complicated in physics by the existence of a multiplicity of different methods for measuring length in accordance with the familiar tendency of a science to extend the meaning of a concept by the assimilation of new methods of measurement as they are discovered. But consideration can be restricted to *length measured by a ruler* (the case of length 'in general' produces no difference in principle) and we can imagine this phrase substituted for *length* in the text.

tion becomes less consistent as very large lengths are reached. It is unnecessary in this context to follow the details of the mathematical treatment of vagueness beyond this sketch of a possible procedure.[36]

We have seen that the relations exhibited in the consistency profile can be regarded as equivalent to a numerical function correlating a numerical value of the consistency to each member x in L's field of application.[37] The consistency curves of L and $\sim L$, or the equivalent numerical functions, constitute the complete analysis of the implications of L and $\sim L$ so far as concerns this vagueness. We eliminate the difficulties due to the inadequacy of the dichotomy of Lx and $\sim Lx$ by providing a more adequate symbolism in which explicit account is taken of those quantitative relations in the field of reference of which the difficulties in interpretation of the dichotomy are a sign.

If the analysis of L, (i.e. the specific consistency profile) be denoted as L', an alternative mode of formulation would be to regard the consistency distribution as indication of the *degree* to which L', the more explicit symbol is applicable to the corresponding terms of the series S. We then regard L in its analysed form as the incomplete expression of a propositional function having *two* arguments, reading $L'(x, C)$ as *$L'$is present in x with degree C.* In this form of expression attention is drawn to the objective relations between L' and S which determine the consistency distribution.

To remove a possible source of misunderstanding it may be as well to add that the analysis of Lx in the manner suggested does not involve the claim that a person asserting Lx in a DxL should know the analysis, i.e. the corresponding distribution of consistencies of application, either at that or at any subsequent time. Any assumption that ability to use a symbol correctly involves extensive statistical knowledge of the behavior of other users would involve a vicious circle. But we can very well use a symbol correctly, i.e. in statistical conformity with the behavior of a certain group of users, without knowing in detail to what we are committed by the linguistic habits of the group.

[36] Cf. Appendix II for further details.

[37] On a frequency theory of probability the assertion of the value of $C(L, x)$ for a given argument x could be interpreted as an assertion concerning the probability of L's application to x. This would involve interpreting every statement of the form Lx as statements of probability lacking a numerical parameter. Such a theory bears a formal resemblance to the theories of say Keynes or Reichenbach (*Wahrscheinlickheitslehre*). But the argument in the text is independent of any particular interpretation of probability.

7. DEFINITION OF THE USERS OF A LANGUAGE

The sense in which language is used in the previous section is clearly a technical one, which needs further discussion.

We need a sense which is narrower than the ordinary sense in which French, Italian or German are called languages. For "In a country like France, Italy or Germany . . . every village or, at most, every group of two or three villages, has its own dialect. . . . The difference from place to place is small but, as one travels in any one direction, the differences accumulate, until speakers, say from opposite ends of the country, cannot understand each other, although there is no sharp line of linguistic demarcation between the places in which they live."[38] Thus "correct" German or "correct" Dutch are better regarded as specially important dialects abstracted from languages which shade into each other by a continuous series of intermediate local variations.[39] We need a sense however which is slightly narrower even than that of the "official" dialect. For if it is a question of obtaining a consistency profile for the name of a color, we shall want to exclude certain persons who cannot speak the official dialect. The observations of the color-blind or those who claim to perceive distinctions in shade invisible to all other persons are valueless in constructing the profile. Thus, without attempting to achieve the empty ideal of strict uniformity we shall find it necessary to exclude from the group of users of the language persons having unusual powers of discrimination. The element of arbitrary convention in the use of the term language in our technical sense is unavoidable but characteristic of all attempts at definition. If however, in view of the enormous variation on geographical, social, technical, and even sexual grounds to be found in lingustic behavior we reject the assumption of *strict* uniformity as too crude an approximation to be useful, it will be necessary to specify how the amount of *permissible* variation is determined. We shall do this by using the process of deriving the consistency profile of a vague symbol simultaneously as a definition of the privileged users of the symbol.

We can begin by considering such a large group of persons that the problem reduces to that of identifying some selection from the group as *the* set of users of the symbol *L*. It has been assumed that *no* set of persons thus selected (not even a set consisting of a single person)

[38] L. Bloomfield, *Language*, 1935, p. 51.
[39] Bloomfield, *cf. cit.*, p. 44.

will show absolute uniformity in applying L. For each set,[40] however, we can apply the procedure of the preceding section, making the number of DxL for each x increase indefinitely, for each x in turn, but keeping the group of observers constant. In this way *some* of the subgroups will provide their own characteristic consistency profiles for the symbol L.

Now the various sub-groups can be classified and amalgamated on the basis of the mutual deviations of their respective consistency profiles. In the simplest case in which the sub-groups separate into a number of non-overlapping classes, each having exactly the same consistency profile, we say that there are as many different usages of L as there are distinct forms of consistency profiles produced by this statistical analysis, and define *the* sub-group of users of L in any one of its meanings as the *largest* sub-group having the corresponding consistency profile.

In practice, however, the situation is likely to be complicated by the existence of a great many consistency profiles with gradations between extreme types. In this case the notion of privileged users whose behavior determines the vagueness of L' will itself be a relatively vague notion. The decision whether to extend any provisionally selected sub-group by the inclusion of new members will depend upon the modifications in the shape of the consistency profiles which such admissions entail. When such modifications are slight the group will be extended, but not otherwise. Newcomers whose admission would entail radical modifications in the shape of the profile will be said to use the terms L and $\sim$L in a different way from that of the group already established. Thus the process of selecting a group of users and of discovering the consistency profile of a symbol are complementary and interact upon each other.

The whole analysis may be compared with the process by which species are defined in biology. There, too, the group of animals constituting a species is defined not by exact correspondence in habits or characters, but by a statistical distribution of variation in habits around a mean position.[41] When a grouping of properties round a mean posi-

[40] Actually to do this would seem to involve examination of the limiting behavior of *each* sub-group. The practical difficulties this would involve and the subsequent modifications required in practice in determining the uses of a language in any specific case need not be considered in a discussion of the principles.

[41] Thus if we adopt a recent definition of species in genetic terms as a "group of individuals fully fertile inter se, but barred from interbreeding with other similar groups by

tion is discovered, it is regarded on the one hand as the *definition* of the species, and on the other as a *property* of the species once it is defined. So too statistical regularity in deviations of linguistic habits is used both to isolate a privileged group and simultaneously to define those habits by the invention of an appropriate symbolism.

APPENDIX I. AN EXPERIMENT IN VAGUENESS

It was hoped that the following experiment might illustrate the formation of a consistency profile and the manner in which consistency of application can be an index of objective relations in the field of reference. Each subject was asked to make a single division, at what seemed "the most natural place," in the series of rectangles shown below, and, having made a unique choice, to analyse the reasons for his decisions. (The instructions used are reproduced at the end of this appendix, so that the results obtained can be checked by others).

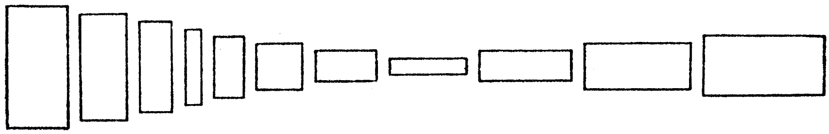

FIG. 4. SERIES OF RECTANGLES USED IN EXPERIMENT ON VAGUENESS

Careful inspection of the series will show at least three[42] criteria for dividing the series which are covered by the deliberately vague word "natural" used in the instructions for the experiment. The rectangles diminish in height by equal steps from left to right until the eighth from the left is reached, and then increase in height again by the same amounts until the end of the row is reached; exactly the same is true of the *breadths* of the rectangles considered in the opposite direction from right to left. In addition to this the heights and breadths are so

its physiological properties (producing either incompatibility of parents or sterility of the hybrids or both)" (T. Dobzhansky, 'Critique of the Species Concept in Biology,' *Phil. of Sci.*, II, 353) we are compelled to admit the qualification that "neither the mechanisms providing incompatibility, nor those producing sterility, function on an all-or-none principle. For instance, sexual isolation may be incomplete, and individuals belonging to different groups may sometimes, though seldom, copulate. Similarly, some hybrids are only semi-sterile or sterile in one sex only" (Ibid.). Nor is the situation fundamentally altered if a genetic definition of this sort is replaced by a taxonomic definition in terms of the possession by the members of the species of certain common characteristics.

[42] In fact, of course, there are an indefinite number of criteria which might be applied; those described are merely the most 'natural' i.e. those which most people tend to use.

correlated that each rectangle is geometrically congruent to another rectangle distant the same number of places from the nearer end of the series, and is obtained from it by a revolution of 180° about an axis perpendicular to the paper. Thus the first and eleventh rectangles, the second and tenth, third and ninth rectangles etc., have the same shape and size. If the criterion of diminishing heights be called *H* for convenience of reference, the criterion of increasing breadth *B* and the criterion of symmetry (which is a kind of combination of *H* and *B*) *S*, it will be seen that application of *H* would result in a decision at a point between *7/8* and *8/9* inclusive (see instructions at end of this appendix for explanation of the notation), application of *B* at a point between *3/4* and *4/5* inclusive, application of S at the Point 6. Thus conflicting criteria produce overlapping fields of application, as in the case of the color spectrum. When the experiment was performed on 83 persons it was found that criterion *B* was seldom employed, the most usual reaction being an application of *H*, *S* or a compromise between the two.[43] Although no particular care was taken to insure homogeneity of the group of subjects, who were in fact of all ages and types, drawn from the writer's acquaintances and students, the corresponding consistency profile, even with the small number on whom the experiment was tried, has quite a distinctive shape of the kind associated with a simply ambiguous symbol. It is to be expected that the characteristic concentration of the number of divisions at or near *6* and *7/8* would be preserved as the number of subjects increased.

Table of experimental results

PLACE AT WHICH DIVISION WAS MADE	NO. OF PERSONS MAKING THE DIVISION AT THAT PLACE
1 to 3/4	None
4	1
4/5	2
5	None
5/6	4
6	36
6/7	6
7	4
7/8	16
8	14
8/9 to 11	None
	Total: 83

[43] Any tendency to make a division in the middle of a series could have been avoided by prolonging both ends of the series a considerable distance or by using a series pasted round a cylinder.

(TYPEWRITTEN INSTRUCTIONS ISSUED WITH THE SERIES OF RECTANGLES)

You are supplied with a series of rectangles arranged in a horizontal line. Do not turn the sheet round; keep them in the horizontal position. You are asked to divide the set of rectangles by a *single* vertical line, at what seems the most NATURAL place. The division may be either between two rectangles or through the middle of one of them. E.g. if all rectangles to the left were red and all the rest were black, the 'natural' place to make the division would be between the red and the black rectangles. *Do not draw any lines*, but show your decision on a separate sheet of paper in the following way:

A division between the third and fourth rectangles from the LEFT is shown by writing 3/4, and so on.

A division through the middle of the tenth rectangle from the LEFT is shown by writing 10, and so on.

You can take as much time as you like, but must make one and only one division.

(The following was issued after the first part of the instructions had been performed):

If you can, try to explain in writing *why* you made the division in the place you did, and add any comments you think interesting (e.g. whether you hesitated between several places).

(Actual dimensions of rectangles: 4 cm. x 2 cm., 3.5 cm. x 1.5 cm., etc. (unit of increase in linear dimension 0.5 cm.). Space between rectangles: 0.5 cm.)

APPENDIX II. EXTENSION OF THE ANALYSIS TO THE LOGICAL RELATIONS BETWEEN VAGUE SYMBOLS

One of the main problems with which this essay has been concerned is the applicability of logical principles when vague symbols are involved. The notion of an ideal universe in which the laws of logic and mathematics have unconfined validity having been rejected, it remains to show how the undoubted usefulness of the formal sciences in a field of vague symbols can be explained by an extension of the method already sketched in the earlier sections.

From the formalist standpoint, the analysis of vagueness in terms of consistency functions can be regarded simply as the introduction of more complex symbolism, replacing the propositional function Lx of a single variable, by a function of two variables, $L(x,C)$, (read: "L applies to x with consistency C"). The relations between symbols in a calculus whose symbols are assumed to be "absolutely precise" will then appear as a limiting case of the relations between symbols having an extra argument, "C", and obtained from the general case by allowing C to tend either to zero or to infinity in every formula in which it occurs, i.e. in effect simply by suppressing that argument.[44] Thus the validity and usefulness of the relations applicable to the limiting case (logical relations between "absolutely precise" symbols) will depend upon the degree to which they can be represented as a standard to which the more general case approximates. In particular, the pure logical relations between the incomplete symbols attained by suppressing parts of vague symbols would appear as limiting cases of relations between vague symbols.

The generalization of the usual notions of material implication or negation of propositional functions of a single variable will be relations connecting the corresponding values of the consistency arguments in two propositional functions. It follows from the definition of the consistency function that if $L(x,C)$ and $\sim L(x,C')$ for the same x, the products of the two consistencies, C and C′, is unity. Thus the principle of excluded middle is replaced by the operation which permits the transformation of $L(x,C)$ into $L\left(x,\frac{1}{C}\right)$.[45]

The consistency profiles of vague symbols can in fact be regarded as a generalization of the circles in the Euler diagrams traditionally used by logicians to represent the relations of inclusion and exclusion between classes.[46] Just as the operation of negating a propositional function corresponds in the usual spatial analogy to the movement from the interior of a circle to its exterior, so also in the spatial illustrations of this paper, the corresponding transition is from L's consistency profile to the reciprocal curve for $\sim L$(cf. Fig. 1).

[44] The limiting process and the suppression of the argument are assumed to produce equivalent effects.

[45] If this principle of transformation is itself formalised (corresponding to the use of the law of excluded middle as a premiss as well as a logical principle) it will be necessary to introduce a further consistency variable. The generalisation of the assertion (X) (Px v $\sim$ Px) will then be

$$(x, c)\ \{P(x, c) \vee \sim P(x, f(c))\}$$

Where f is some specified function of c (nearly equal to c when C is near to 1, very small when c is large and very large when c is small. The exact form of $f(c)$ would depend on the exact form of the consistency curves in the special case.)

[46] Since such diagrams habitually assume a two-dimensional field of application, the corresponding diagram of consistencies of application should strictly be three-dimensional and consist of a (polyhedral) surface obtained by joining the top of adjacent ordinates erected not upon an axis (OS in Fig. 2) but upon a plane of reference. The argument of the section can, however, be sufficiently illustrated by supposing that the field of application is a one-dimensional series as in Figs. 1–3 above.

This in turn suggests a generalization for the relation of implication between propositional functions, or the equivalent relation of inclusion between their extensions. We take as the *type* of the case when *L*'s field of application includes *M*'s field the case in which *M*'s consistency profile lies wholly *underneath* *L*'s consistency profile, i.e. for every x, *M*'s consistency of application is less than *L*'s consistency of application to that x. It is necessary to say that this definition specifies only the type or standard case of the relationship between two functions, because, in accordance with the general standpoint of this paper, deviations from the type must be allowed. If the consistency of M is greater than the consistency of L in *very few* of the x's, we shall still say $Mx \supset Lx$ to *some* extent. We can imagine a number $i(L,M)$ which measures the degree to which the relation between the consistency profiles of L and M deviates from the standard case of inclusion.[47] Then i, which might be called an approximation index for the relation of inclusion is a function of the consistency distributions of L and M.

Thus the statement that $Mx \supset Lx$ will be generalized into some such form as

$$\supset\{i(L,M),\ c\}$$

i.e. a propositional function of two variables, viz. the approximation index for inclusion, and the usual consistency variable. Hence the syllogistic law which permits of transition from the two formulae

$$\supset(L,M) \text{ and } \supset(M,N)$$

to the formula

$$\supset(L,N)$$

is generalized into some rule connecting the different approximation indices in

$$\supset\{i(L,M),\ c\},\ \supset\{i'(M,N),\ c\}$$

and

$$\supset\{i''(L,N),\ c\}$$

Thus if the approximation index were suitably defined it might follow that i'' must be $< i + i'$. The rule for passing from $\supset(L,M)$ and

[47] i(L, M) might be, for example, the ratio of the number of x's when *M*'s consistency is greater than *L*'s to the number when *M*'s consistency is less than *L*'s. The exact definition which is chosen is unimportant.

$\supset(M,N)$ to $\supset(L,N)$ would then become: Whenever $\supset\{i(L,M), c\}$ and $\supset\{i'(M,N), c\}$ are asserted for the same value of *c* and specific (but not necessarily identical) values of i and i′, then $\supset\{i''(L,N), c\}$ can be asserted, with the same value of c, the value of i″ certainly being less than the sum of i and i′. In the special case where $i = i' = 0$ (strict inclusion) we shall have $i'' = 0$, and the ordinary relation will hold.

We can characterize the preceding interpretation very roughly in this way: the ordinary rules for the logical transformation of sets of statements (e.g. the syllogistic rules) produce conclusions whose degree

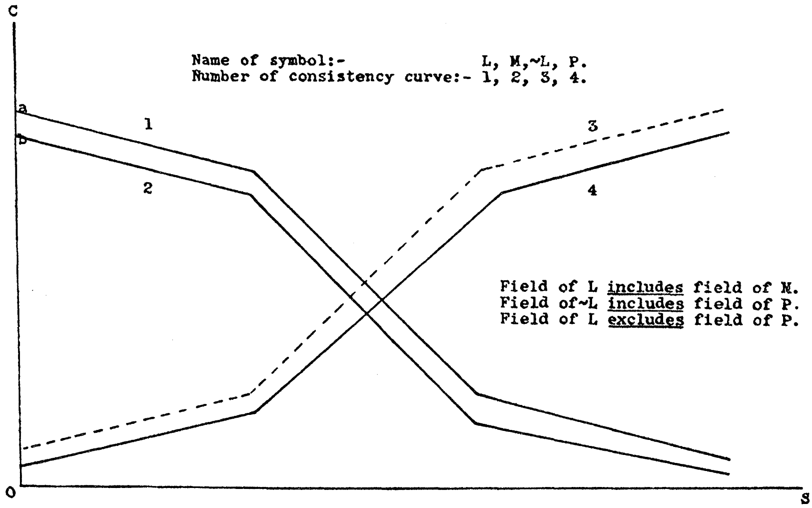

Fig. 5. Logical Relations between Vague Symbols Spatially Illustrated

of vagueness is of the same order[48] as those of the premisses. In proportion as the relations between the consistency curves approximate to the definitions of inclusion and negation given above, the resulting laws of logical transformation approximate to those of the traditional forms.

By means of the notions of inclusion and negation it is easy to define the relation of *exclusion* between consistency functions. We say that *L*'s field *excludes P*'s if $\sim$*L*'s field includes *P*'s. The standard cases of the relationships arising are shown in the figure below. It is easily verified that the new definitions of inclusion, exclusion and negation

[48] This means roughly speaking that it tends to zero when the degree of vagueness of all the premises tend to zero.

preserve the usual formal properties of these relationships. Thus both inclusion and exclusion remain transitive: also if *L*'s field includes *M*'s, $\sim$*M*'s field includes $\sim$*L*'s etc. (See Fig. 5 on preceding page.)

Thus, if the logical forms are interpreted as standard cases to which the relations between consistency functions may approximate, the formal properties on which the theory of formal logic are based remain valid in the approximatory sense discussed.

For while the vagueness not only in the terms of the premisses but in their relations prevents us from asserting the conclusion of an argument in applied logic or applied mathematics without a qualification as to the degree of consistency (whose amount depends on the precision of the terms and logical relations) the *form* of the transformation is independent of the actual consistencies provided we are satisfied with a final precision which increases indefinitely when the precision of the premisses increases.

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