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FUZZY HIERARCHICAL ANALYSIS*

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This paper extends hierarchical analysis to the case where the participants are allowed to employ fuzzy ratios in place of exact ratios. If a person considers alternative A more important than alternative B, then the ratio used might be approximately 3 to 1, or between 2 to 1, and 4 to 1, or at most 5 to 1. The pairwise comparison of the issues and the criteria in the hierarchy produce fuzzy positive reciprocal matrices. The geometric mean method is employed to calculate the fuzzy weights for each fuzzy matrix, and these are combined in the usual manner to determine the final fuzzy weights for the alternatives. The final fuzzy weights are used to rank the alternatives from highest to lowest. The highest ranking contains all the undominated issues. The procedure easily extends to the situation where many experts are utilized in the ranking process, or to the case of missing data. Two examples are presented showing the final fuzzy weights and the final ranking.

Keywords: Decision making, Multicriteria analysis.

1. Introduction

This paper develops Saaty's hierarchical analysis [7, 8, 9], when the experts (judges, ...) are allowed to use fuzzy ratios in place of exact ratios. In Saaty's hierarchical analysis a person (expert, judge, ...) is asked to supply ratios a_{ij} for each pairwise comparison between issues (alternatives, candidates, ...) A_1, $A_2, \dots, A_m$ for each criterion (objective) in a hierarchy, and also between the criteria. For some specific criterion, if a person considers A_1 more important than A_5, then a_{15} might equal 3/1, or 5/1, or 7/1. The numbers for the ratios are usually taken from the set $\{1, 2, \dots, 9\}$ so a_{15} could be s_1/s_5 for s_1, $s_5 \in \{1, 2, \dots, 9\}$ and $s_1 > s_5$. The ratios a_{ij} indicate, for this expert, the strength with which A_i dominates A_j. If a_{15} is equal to 5/1, then a_{51} is taken as 1/5. That is, $a_{ji} = a_{ij}^{-1}$ and $a_{ii} = 1$ for all i. Let A be the $m \times m$ matrix whose entries are the ratios a_{ij}. A is called a positive reciprocal matrix. Saaty's procedure uses the pairwise comparison matrices A for each criterion, and also the pairwise comparison matrix for the criteria, to compute a final set of weights w_i ($w_i > 0$, $w_1 + \cdots + w_m = 1$) for the alternatives which can be used to rank the issues from highest to lowest.

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In this paper the a_{ij} can be fuzzy numbers [2–5]. Fuzzy numbers can express an expert's feeling that a ratio a_{ij} is approximately 5 to 1 instead of exactly 5/1, or that a ratio is between 6 to 1 and 8 to 1 instead of exactly 7/1. We all recognize that it is difficult for people to always assign exact ratios when comparing two alternatives. When comparing A_1 and A_5 a person might feel that A_1 is much more important than A_5. Does this mean that a_{15} equals 7/1, or 8/1, or 9/1? Using fuzzy numbers the expert can respond that a_{15} is between 7 to 1 and 9 to 1. Also, a person could feel that A_1 is a little more important than A_5. An appropriate fuzzy ratio in this case might be approximately 3 to 1. Fuzzy numbers automatically incorporate the vagueness of these replies.

In the next section we introduce the fuzzy numbers which experts will be allowed to use in pairwise comparisons and we discuss consistency of the data. The following section investigates methods of deriving the final fuzzy weights for the issues and discusses a procedure for ranking the alternatives given their fuzzy weights. The fourth section presents various properties of fuzzy hierarchical analysis. The following section extends the technique to the situation where input from many experts is desired. The relationship of this paper to previous research [13, 14] is discussed in the sixth section. Two examples are worked out in the following section and the last section contains a summary and our conclusions.

2. Fuzzy ratios and consistency

The type of fuzzy numbers used (see [2]) by the experts in paired comparisons is described by $(\alpha/\beta, \gamma/\delta)$ where $0<\alpha\leqslant\beta\leqslant\gamma\leqslant\delta$. The graph of the membership function μ is determined by these four numbers as follows: zero to the left of α, a straight line segment from $(\alpha, 0)$ to $(\beta, 1)$, a horizontal line segment from $(\beta, 1)$ to $(\gamma, 1)$, a straight line segment from $(\gamma, 1)$ to $(\delta, 0)$, and zero to the right of δ. From now on we will place a bar over a symbol if it represents fuzzy numbers. If an expert believes alternative A_i is more important than A_j, then the fuzzy ratio $\bar{a}_{ij}=(\alpha_{ij}/\beta_{ij}, \gamma_{ij}/\delta_{ij})$ has α, β, γ, $\delta\in\{1, 2, \ldots, 9\}$ and $\bar{a}_{ji}$ is taken as $(\bar{a}_{ij})^{-1}=(\delta_{ij}^{-1}/\gamma_{ij}^{-1}, \beta_{ij}^{-1}/\alpha_{ij}^{-1})$. For the reciprocal of a fuzzy number see [5], page 49.

If two of the numbers α, β, or β, γ, or γ, δ are equal in a fuzzy number $\bar{a}_{ij}$, then the corresponding line segment does not exist. For example, (4/5, 5/6) is a triangle over the interval [4, 6] and (2/2, 2/4) is a line segment from (2, 1) to (4, 0) and zero elsewhere. Any real numer n is equal to $(n/n, n/n)$. Therefore, if a person feels that A_i and A_j are equally important, then $\bar{a}_{ij}=(1/1, 1/1)$. Let $\bar{A}$ be the $m\times m$ fuzzy matrix of all paired comparisons for the issues $A_1, A_2, \ldots, A_m$. The elements in $\bar{A}$ are $\bar{a}_{ij}$ where $\bar{a}_{ji}=(\bar{a}_{ij})^{-1}$ and $\bar{a}_{ii}=(1/1, 1/1)$ all i. $\bar{A}$ is called a fuzzy positive reciprocal matrix.

The fuzzy numbers are intuitively easy to use and have various interpretations which would be explained to a person before they would be asked to suply the fuzzy positive reciprocal matrices $\bar{A}$. For example, approximately 5 to 1 might be (4/5, 5/6) and between 6 to 1 and 8 to 1 could be (6/6, 8/8) or (5/6, 8/9). Also, one mght interpret (3/3, 3/5) as being at least 3 to 1.

We now need to introduce a method of comparing fuzzy numbers that will be used in the definition of consistency and in a later setion when we want to rank the alternatives from highest to lowest. Let $\bar{M}$ and $\bar{N}$ be two fuzzy numbers with membership functions $\mu_m(x)$ and $\mu_n(x)$, respectively. An arbitrary fuzzy number $\bar{M}$ is also determined by the numbers $(m_1/m_2, m_3/m_4)$ but the graph of $\mu_m(x)$ need not be a straight line segment on $[m_1, m_2]$ and $[m_3, m_4]$. The membership function $\mu(x)$ is continuous monotonically increasing from zero to one on $[m_1, m_2]$ and continuous monotonically decreasing from one to zero on $[m_3, m_4]$. Define

$$v(\bar{M} \geqslant \bar{N}) = \sup_{x \geqslant y}(\min(\mu_m(x), \mu_n(y))).$$

We will say that $\bar{M}$ is greater than $\bar{N}$, written $\bar{M} > \bar{N}$, if $v(\bar{M} \geqslant \bar{N}) = 1$ and $v(\bar{N} \geqslant \bar{M}) < \theta$, where θ is some fixed positive fraction less than or equal to one [4; 5, p. 59–60]. Values like 0.7, 0.8, 0.9 might be appropriate for θ. If $\bar{M}$ is not greater than $\bar{N}$ and $\bar{N}$ is not greater than $\bar{M}$, we will say $\bar{M}$ and $\bar{N}$ are approximately equal which is written as $\bar{M} \approx \bar{N}$. Therefore, if

$$\min(v(\bar{M} \geqslant \bar{N}), v(\bar{N} \geqslant \bar{M})) \geqslant \theta,$$

then $\bar{M} \approx \bar{N}$. We will write $\bar{M} \geqslant \bar{N}$, when $\bar{M} > \bar{N}$ or $\bar{M} \approx \bar{N}$. We will also need to add and multiply fuzzy numbers. Let $\oplus$ and $\odot$ be the standard fuzzy addition and multiplication, respectively [3; 4; 5, Chapter 3].

Definition 1. A fuzzy positive reciprocal matrix $\bar{A} = [\bar{a}_{ij}]$ is consistent if and only if $\bar{a}_{ik} \odot \bar{a}_{kj} \approx \bar{a}_{ij}$.

This definition is the direct extension to fuzzy positive reciprocal matrices of the definition of (ordinal) consistency for positive reciprocal matrices [7, 9]. If $A = [a_{ij}]$ is a positive reciprocal matrix, then A is consistent if and only if $a_{ik}a_{kj} = a_{ij}$.

Much of fuzzy hierarchical analysis will reduce to Saaty's hierarchial analysis if all the fuzzy numbers are triangular $(\beta_{ij} = \gamma_{ij})$ and $\theta = 1$. $\bar{A}$ is consistent if and only if $\beta_{ik}\beta_{kj} = \beta_{ij}$ when $\beta_{ij} = \gamma_{ij}$ and $\theta = 1$. If some flat fuzzy numbers are employed $(\beta_{ij} < \gamma_{ij})$ and/or $\theta < 1$, then fuzzy hierarchical analysis is an extension of classical hierarchical analysis.

If a fuzzy positive reciprocal matrix is not consistent we may wish to revise the estimates of the ratios so that the estimates of the fuzzy weights may be improved. Saaty [7] states:

> Note that improving consistency does not mean getting an answer closer to the 'real' life solution. It only means that the ratio estimates in the matrix, as a sample collection, are closer to being logically related than to being randomly chosen.

Theorem 1. *Let $\bar{A} = [\bar{a}_{ij}]$ where $\bar{a}_{ij} = (\alpha_{ij}/\beta_{ij}, \gamma_{ij}/\delta_{ij})$ and let $\beta_{ij} \leqslant a_{ij} \leqslant \gamma_{ij}$ for all i, j. If $A = [a_{ij}]$ is consistent, then $\bar{A}$ is consistent.*

Proof. We first determine $\bar{a}_{ik} \odot \bar{a}_{kj}$ (see [1]). The graph of the membership function for this fuzzy number is zero to the left of $\alpha_{ik}\alpha_{kj}$, monotonically increases from $(\alpha_{ik}\alpha_{kj}, 0)$ to $(\beta_{ik}\beta_{kj}, 1)$ on the interval $[\alpha_{ik}\alpha_{kj}, \beta_{ik}\beta_{kj}]$, is a horizontal line segment between $(\beta_{ik}\beta_{kj}, 1)$ and $(\gamma_{ik}\gamma_{kj}, 1)$, monotonically decreases from $(\gamma_{ik}\gamma_{kj}, 1)$ to $(\delta_{ik}\delta_{kj}, 0)$ on the interval $[\gamma_{ik}\gamma_{kj}, \delta_{ik}\delta_{kj}]$, and is zero to the right of $\delta_{ik}\delta_{kj}$. The increasing and decreasing parts are not straight line segments (see [1]). If two of the numbers $\alpha_{ik}\alpha_{kj}, \beta_{ik}\beta_{kj}$, or $\beta_{ik}\beta_{kj}, \gamma_{ik}\gamma_{kj}$, or $\gamma_{ik}\gamma_{kj}, \delta_{ik}\delta_{kj}$ are equal, then that part of the graph does not exist.

Since $\beta_{ik}\beta_{kj} \leqslant a_{ik}a_{kj} = a_{ij} \leqslant \gamma_{ik}\gamma_{kj}$ with $\beta_{ij} \leqslant a_{ij} \leqslant \gamma_{ij}$, we see that

$$v(\bar{a}_{ik} \odot \bar{a}_{kj} \geqslant \bar{a}_{ij}) = 1 \quad \text{and} \quad v(\bar{a}_{ij} \geqslant \bar{a}_{ik} \odot \bar{a}_{kj}) = 1.$$

Therefore, $\bar{a}_{ik} \odot \bar{a}_{kj} \approx \bar{a}_{ij}$, and $\bar{A}$ is consistent.

The converse of Theorem 1 is not necessarily true because the intervals where the membership function for $\bar{a}_{ij}$ and $\bar{a}_{ik} \odot \bar{a}_{kj}$ is equal to one may be disjoint when $\theta < 1$. Even if $\theta = 1$, the converse is not necessarily true.

3. The fuzzy weights

In Saaty's hierarchical analysis one first computes the weights w_i ($w_i > 0$, $w_1 + \cdots + w_m = 1$) for each positive reciprocal matrix A. Then these weights are combined, depending on the hierarchical structure, to obtain the final set of weights for the alternatives. Therefore, we must first specify how to obtain the fuzzy weights $\bar{w}_i$ given any fuzzy positive reciprocal matrix $\bar{A}$.

We will begin with Saaty's λ-max procedure for determining the weights and show that it is not readily extended to fuzzy matrices. For a given A, λ-max is the largest (real) eigenvalue of A and the weights w_i are the components of the normalized (sum equals one) eigenvector corresponding to λ-max. Now consider a fuzzy positive reciprocal matrix $\bar{A} = [\bar{a}_{ij}]$ where $\bar{a}_{ij} = (\alpha_{ij}/\beta_{ij}, \gamma_{ij}/\delta_{ij})$. Generalizing the λ-max method we would consider

$$\bar{A} \odot \bar{w} = \bar{\lambda} \odot \bar{w}$$

where $\bar{w}^{\mathrm{T}} = (\bar{w}_1, \ldots, \bar{w}_m)$ and the $\bar{w}_i$ and $\bar{\lambda}$ are fuzzy numbers. The above equation defines the following m equations

$$(\bar{a}_{i1} \odot \bar{w}_1) \oplus \cdots \oplus (\bar{a}_{im} \odot \bar{w}_m) = \bar{\lambda} \odot \bar{w}_i.$$

The fuzzy numbers $\bar{w}_i$ are determined by $(\varepsilon_i/\xi_i, \eta_i/\theta_i)$ where the graph of the membership function μ_i is zero to the left of ε_i, continuous and monotonically increasing from $(\varepsilon_i, 0)$ to $(\xi_i, 1)$, a horizontal line segment from $(\xi_i, 1)$ to $(\eta_i, 1)$, continuous and monotonically decreasing from $(\eta_i, 1)$ to $(\theta_i, 0)$, and zero to the right of θ_i. Similarly, the fuzzy number $\bar{\lambda}$ is determined by $(\lambda_1/\lambda_2, \lambda_3/\lambda_4)$.

Let $A = [\alpha_{ij}]$, $B = [\beta_{ij}]$, $C = [\gamma_{ij}]$ and $D = [\delta_{ij}]$. Also, let $X^1 = (\varepsilon_1, \ldots, \varepsilon_m)^{\mathrm{T}}$, $X^2 = (\xi_1, \ldots, \xi_m)^{\mathrm{T}}$, $X^3 = (\eta_1, \ldots, \eta_m)^{\mathrm{T}}$, and $X^4 = (\theta_1, \ldots, \theta_m)^{\mathrm{T}}$. The above system of equations implies that $AX^1 = \lambda_1 X^1$, $BX^2 = \lambda_2 X^2$, $CX^3 = \lambda_3 X^3$, $DX^4 = \lambda_4 X^4$. In each of these equations let λ_i-max be the largest (real) eigenvalue and V^i the associated normalized eigenvector. Therefore, V^1 would contain the values of the

ε_i, V^2 has the values of the ξ_i, etc. Since the components of each V^i sum to one we cannot have $\varepsilon_i \leq \xi_i \leq \eta_i \leq \theta_i$ for all i. Since the normalized eigenvectors cannot be used to define $\bar{w}$, the question is: "what eigenvector should be used?" There appears to be no reason to choose one eigenvector over another so we conclude that this method cannot be employed to define the largest fuzzy eigenvalue and a corresponding fuzzy eigenvector of weights. One might next consider $\bar{A} \odot \bar{w} \approx \bar{\lambda} \odot \bar{w}$, but we shall not pursue this direction in this paper.

The geometric mean technique [1, 11, 12] for computing the weights w_i is easily extended to fuzzy positive reciprocal matrices $\bar{A}$. Given a positive reciprocal matrix $A = [a_{ij}]$, first compute the geometric mean of each row as

$$r_i = \left(\prod_{j=1}^{m} a_{ij} \right)^{1/m},$$

and then $w_i = r_i/(r_1 + \cdots + r_m)$. If A is consistent, then the geometric mean method always produces the same weights as Saaty's λ-max technique and if $m = 3$, both methods compute the same weights [10]. It appears [11, 12] that when $m > 3$ the numerical results for the weights in the two procedures are close to each other.

Other methods have been proposed for estimating the true weights from a given positve reciprocal matrix. It is not our intention to get involved in the debate over which procedure is best. We wish to choose that method which extends easily to fuzzy positive reciprocal matrices and possesses a number of desirable properties. We believe it is the log least squares method that results in the geometric mean procedure.

For $\bar{A} = [\bar{a}_{ij}]$ define

$$\bar{r}_i = (\bar{a}_{i1} \odot \cdots \odot \bar{a}_{im})^{1/m} \quad \text{and} \quad \bar{w}_i = \bar{r}_i \odot (\bar{r}_1 \oplus \cdots \oplus \bar{r}_m)^{-1}.$$

Roots of fuzzy numbers are discussed in [5], page 53. In the remainder of this paper we will employ this technique for computing the fuzzy weights $\bar{w}_i$.

Based on [2] we will determine the membership function μ_i for $\bar{w}_i$. Let

$$f_i(y) = \left[\prod_{j=1}^{m} ((\beta_{ij} - \alpha_{ij})y + \alpha_{ij}) \right]^{1/m},$$

$$g_i(y) = \left[\prod_{j=1}^{m} ((\gamma_{ij} - \delta_{ij})y + \delta_{ij}) \right]^{1/m},$$

for $0 \leq y \leq 1$. Define $\alpha_i = [\prod_{j=1}^{m} \alpha_{ij}]^{1/m}$ and $\alpha = \sum_{i=1}^{m} \alpha_i$. Similarly, define β_i and β, γ_i and γ, δ_i and δ. Finally, let

$$f(y) = \sum_{i=1}^{m} f_i(y), \qquad g(y) = \sum_{i=1}^{m} g_i(y).$$

The fuzzy weights $\bar{w}_i$ are determined by $(\alpha_i\delta^{-1}/\beta_i\gamma^{-1}, \gamma_i\beta^{-1}/\delta_i\alpha^{-1})$ where the graph of μ_i is zero to the left of $\alpha_i\delta^{-1}$, $x = f_i(y)/g(y)$ on the interval $[\alpha_i\delta^{-1}, \beta_i\gamma^{-1}]$, a horizontal line from $(\beta_i\gamma^{-1}, 1)$ to $(\gamma_i\beta^{-1}, 1)$, $x = g_i(y)/f(y)$ on the interval $[\gamma_i\beta^{-1}, \delta_i\alpha^{-1}]$, and zero to the right of $\delta_i\alpha^{-1}$. We have assumed the x-axis is horizontal and the y-axis is vertical so the graph of $x = f_i(y)/g(y)$ starts at

$(\alpha_i\delta^{-1}, 0)$ for $y=0$ and monotonically increases to $(\beta_i\gamma^{-1}, 1)$ as y grows from 0 to 1. One might wish to multiply each $\bar{w}_i$ by a normalizing constant so that the support of the $\bar{w}_i$ (where $\mu_i(x)>0$) lies in the interval [0, 1].

Theorem 2 (Preservation of ordinal consistency [7]). *Suppose $\bar{a}_{ij} > \bar{a}_{kj}$ for $j = 1, \ldots, m$. Then $\bar{w}_i \geqslant \bar{w}_k$.*

Proof. If $\bar{w}_i = (\varepsilon_i/\xi_i, \eta_i/\theta_i)$, then the result follows from $\eta_i \geqslant \xi_k$. Now

$$\eta_i = \left(\prod_{j=1}^{m} \gamma_{ij}\right)^{1/m} \Big/ \beta \quad \text{and} \quad \xi_k = \left(\prod_{j=1}^{m} \beta_{kj}\right)^{1/m} \Big/ \gamma.$$

We have $1/\beta \geqslant 1/\gamma$ because $\beta \geqslant \gamma$. The inequality $\bar{a}_{ij} > \bar{a}_{kj}$ all j implies $\beta_{ij} > \gamma_{kj}$ all j, which implies $\gamma_{ij} > \beta_{kj}$ all j. It follows that $\eta_i \geqslant \xi_k$ and $\bar{w}_i \geqslant \bar{w}_k$.

The procedure outlined above determines the fuzzy weights for any fuzzy positive reciprocal matrix $\bar{A}$. To obtain the final fuzzy weights, and the final ranking of the issues, we need to consider a specific hierarchy. To illustrate our method consider the structure in Figure 1. For each criterion C_k (aspect, characteristic, ...) we obtain a fuzzy positive reciprocal matrix $\bar{A}_k$ of pairwise comparisons. We also obtain a fuzzy positive reciprocal matrix $\bar{E}$ for the pairwise comparison of the criteria. Fuzzy weights $\bar{w}_k = (\bar{w}_{1k}, \ldots, \bar{w}_{mk})$ are computed for each $\bar{A}_k$ and fuzzy weights $\bar{e} = (\bar{e}_1, \ldots, \bar{e}_K)$ are derived from $\bar{E}$. The final fuzzy weight for issue A_j is

$$\bar{f}_j = (\bar{w}_{j1} \odot \bar{e}_1) \oplus \cdots \oplus (\bar{w}_{jK} \odot \bar{e}_K).$$

The membership functions for the $\bar{f}_j$ are easily found from the membership

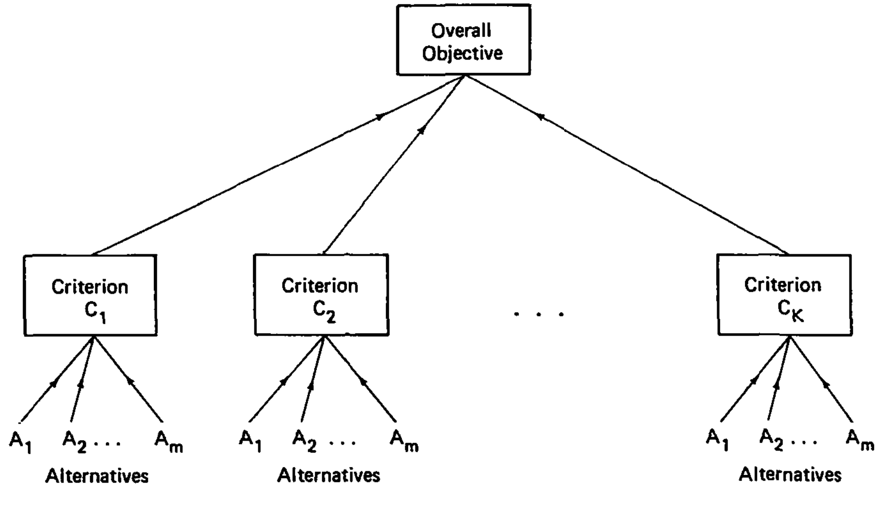

Fig. 1. Hierarchical structure.

functions for the $\bar{w}_{jk}$ and $\bar{e}_j$. Again, one might multiply each $\bar{f}_j$ by a suitable constant so that all the $\bar{f}_j$ have their support in [0, 1].

Definition 2. Alternative A_i dominates issue A_j, written $A_i > A_j$, if and only if $\bar{f}_i > \bar{f}_j$. If A_i does not dominate A_j and A_j does not dominate A_i, we write $A_i \approx A_j$.

Define the sets H_1, $H_2, \ldots$ as follows. H_1 contains all the undominated issues. H_2 has all the undominated alternatives after deleting all the A_i in H_1. Similarly, we define $H_3, H_4, \ldots$ with the last non-empty set called H_d. Clearly, $H_1 \neq \emptyset$, and as long as H_1 does not contain all the alternatives, H_2 is non-empty, etc. All the issues in an H_j are at the same level in the ranking. All the A_i in H_1 have the highest ranking, the issues in H_2 are all at the second highest level, etc. The final ranking of the $A_1, A_2, \ldots, A_m$ is accomplished by the sets $H_1, H_2, \ldots, H_d$.

4. Properties of fuzzy hierarchical analysis

We write $A_i \geqslant A_j$ if $A_i > A_j$ or $A_i \approx A_j$. The following properties of $>$, $\geqslant$, and $\approx$ are easily verified and so the proofs are omitted [2].

(a) $>$ is transitive, $\approx$ is not transitive.

(b) Given any two issues A_i and A_j, then one of the following holds: $A_i > A_j$, or $A_i \approx A_j$, or $A_j > A_i$.

(c) If A_i and A_j belong to H_k, then $A_i \approx A_j$.

(d) Given any $A_i \in H_k$, $k > 1$, there is an $A_j \in H_{k-1}$ so that $A_j > A_i$.

(e) $\geqslant$ is transitive, anti-symmetric, and reflexive (a partial order).

(f) If $A_i \in H_k$, then $A_i \geqslant A_j$ for all $A_j \in H_l$, $l \geqslant k$.

(g) $\geqslant$ is a total order (partial order plus (b) above).

We next observe that fuzzy hierarchical analysis 'contains' standard hierarchical analysis when the geometric mean procedure is employed to calculate the weights. As before, $\bar{A}_k = [\bar{a}_{ij}^k]$, $\bar{a}_{ij}^k = (\alpha_{ij}^k/\beta_{ij}^k, \gamma_{ij}^k/\delta_{ij}^k)$, are the fuzzy positive reciprocal matrices for the criteria C_k, $k = 1, 2, \ldots, K$, and $\bar{E} = [\bar{e}_{ij}]$, $\bar{e}_{ij} = (\pi_{ij}/\rho_{ij}, \sigma_{ij}/\tau_{ij})$, is the pairwise comparison matrix for the criteria. Let $a_{ij}^k \in [\beta_{ij}^k, \gamma_{ij}^k]$ for all i, j, k and let $b_{ij} \in [\rho_{ij}, \sigma_{ij}]$ for all i, j. Form the matrices $M_k = [a_{ij}^k]$ and $M = [b_{ij}]$. Use the geometric mean method to find the weights for the matrices M_k and M and then combine them in the usual way to obtain the final weights f_i for the alternatives. If $\bar{f}_i = (\phi_i/\chi_i, \psi_i/\omega_i)$ are the fuzzy weights obtained from the $\bar{A}_k$ and $\bar{E}$, then it is not difficult to see that $f_i \in [\chi_i, \psi_i]$ for all i. That is, the (non-fuzzy) weights f_i belong to the interval where the membership function for $\bar{f}_i$ equals one.

Suppose in a pairwise comparison of the issues $A_1, \ldots, A_m$ the 'true' weights are $w_1^*, w_2^*, \ldots, w_m^*$, where $w_i^* > 0$ and $w_1^* + \cdots + w_m^* = 1$. If an expert knew the w_i^*, then all the a_{ij} in the pairwise comparison matrix A would equal w_i^*/w_j^* and A would be consistent. In practice, the $\bar{a}_{ij}$ used are considered estimates of w_i^*/w_j^* since the true weights are usually not kown.

Theorem 3. *Let $\bar{A} = [\bar{a}_{ij}]$, $\bar{a}_{ij} = (\alpha_{ij}/\beta_{ij}, \gamma_{ij}/\delta_{ij})$ and suppose $w_i^*/w_j^* \in [\beta_{ij}, \gamma_{ij}]$ for all i, j. Then $\bar{A}$ is consistent and $w_i^* \in [\xi_i, \eta_i]$ for all i, where $\bar{w}_i = (\varepsilon_i/\xi_i, \eta_i/\theta_i)$.*

Proof. The fact that $\bar{A}$ is consistent follows from Theorem 1. Let $M=[w_i^*/w_j^*]$. It follows from the discussion above that the weights w_i obtained from M via the geometric mean procedure belong to the corresponding interval $[\xi_i, \eta_i]$. But $w_i = r_i^*/(r_1^*+\cdots+r_m^*)$, where

$$r_i^* = \left[\prod_{j=1}^{m} (w_i^*/w_j^*)\right]^{1/m}.$$

It follows that $w_i^* = w_i$ because the sum of the w_i^* is one.

Theorem 2 states that if the true ratios belong to the intervals where the membership function for $\bar{a}_{ij}$ is one, then the true weights belong to the intervals where the membership functions for $\bar{w}_i$ equals one.

It is of interest to speculate when all the fuzzy weights will be the same. Suppose an expert believes that A_1 is more important than A_2, A_2 is more important than $A_3, \ldots,$ and A_m is more important than A_1. The expert's preferences are not transitive and the fuzzy weights $\bar{w}_i$ will depend on the other comparisons between A_1 and A_3, A_2 and A_4, etc. When there are only three issues we now show the fuzzy weights must all be equal. This situation is sometimes called the voter's paradox, or Arrow's paradox. The proof of the following theorem is straightforward and hence omitted.

Theorem 4. *Suppose $m=3$ and $\bar{a}_{12}=\bar{a}_{23}=\bar{a}_{31}$. Then $\bar{w}_1=\bar{w}_2=\bar{w}_3$.*

5. Multiple experts

Multiple experts are now employed in the hierarchical analysis in order to rank the alternatives. Other authors [1, 12] have used the geometric mean to aggregate, or pool, all the data across the experts before computing the weights from the 'average' matrix.

Suppose the experts (judges, ...) are called $J_1, \ldots, J_n$. Each judge J_l supplies a fuzzy positive reciprocal matrix $\bar{A}_{kl}$ for each criterion C_k in the hierarchy, and J_l also produces a matrix $\bar{E}_l$ of paired comparisons between the criteria. Let $\bar{A}_{kl}=[\bar{a}_{ij}^{kl}]$ and $\bar{E}_l=[\bar{e}_{ij}^{l}]$. The average fuzzy positive reciprocal matrices $\bar{A}_k=[\bar{a}_{ij}^{k}]$ and $\bar{E}=[\bar{e}_{ij}]$ are determined as follows:

$$\bar{a}_{ij}^{k}=(\bar{a}_{ij}^{k1}\odot\cdots\odot\bar{a}_{ij}^{kn})^{1/n}, \qquad \bar{e}_{ij}=(\bar{e}_{ij}^{1}\odot\cdots\odot\bar{e}_{ij}^{n})^{1/n}.$$

It may be checked that $(\bar{a}_{ij}^{k})^{-1}=\bar{a}_{ji}^{k}$ and $(\bar{e}_{ij})^{-1}=\bar{e}_{ji}$.

Theorem 5. *If $\bar{A}_{k1}, \ldots, \bar{A}_{kn}$ are consistent, then $\bar{A}_k$ is consistent.*

Proof. The proof follows from the following three lemmas. $\bar{M}$, $\bar{N}$, $\bar{P}$, $\bar{Q}$ are fuzzy numbers. $\bar{M}$ is determined by $(m_1/m_2, m_3/m_4)$ for $0<m_1\leqslant m_2\leqslant m_3\leqslant m_4$. The graph of the membership function $\mu_m(x)$ for $\bar{M}$ is continuous and monotonically increasing on $[m_1, m_2]$ and continuous and monotonically decreasing on $[m_3, m_4]$.

The other fuzzy numbers $\bar{N}$, $\bar{P}$, $\bar{Q}$ are defined the same way and are determined by $(n_1/n_2, n_3/n_4)$, $(p_1/p_2, p_3/p_4)$, $(q_1/q_2, q_3/q_4)$, respectively.

Lemma 1. $(\bar{M}\odot\bar{N})^{1/s}\odot(\bar{P}\odot\bar{Q})^{1/s}=[(\bar{M}\odot\bar{N})\odot(\bar{P}\odot\bar{Q})]^{1/s}$, $s=2,3,4,\ldots$.

Proof. The fuzzy number on the left side of the equation is determined by

$$((m_1n_1)^{1/s}(p_1q_1)^{1/s}/(m_2n_2)^{1/s}(p_2q_2)^{1/s}, (m_3n_3)^{1/s}(p_3q_3)^{1/s}/(m_4n_4)^{1/s}(p_3q_3)^{1/s}).$$

The fuzzy number on the right of the equality is determined by

$$((m_1n_1p_1q_1)^{1/s}/(m_2n_2p_2q_2)^{1/s}, (m_3n_3p_3q_3)^{1/s}/(m_4n_4p_4q_4)^{1/s}).$$

Therefore, these two expressions are equal. One may also check that the increasing and decreasing parts of the graphs are also equal.

Lemma 2. *If $\bar{M}\approx\bar{P}$ and $\bar{Q}\approx\bar{N}$, then $\bar{M}\odot\bar{N}\approx\bar{P}\odot\bar{Q}$.*

Proof. The proof is easy if $\theta=1$ because then $[m_2,m_3]\cap[p_2,p_3]\neq\emptyset$ and $[n_2,n_3]\cap[q_2,q_3]\neq\emptyset$. So assume that $\theta<1$. We may also assume that $m_3n_3<p_2q_2$ or $p_3q_3<m_2n_2$ for otherwise $[m_2n_2,m_3n_3]\cap[p_2q_2,p_3q_3]\neq\emptyset$ and $\bar{M}\odot\bar{N}\approx\bar{P}\odot\bar{Q}$. Both arguments are the same so consider $m_3n_3<p_2q_2$. Let $b_1\in[p_1,p_2]$ and $b_2\in[q_1,q_2]$ so that $\mu_p(b_1)=\theta$ and $\mu_q(b_2)=\theta$. Also let $a_1\in[m_3,m_4]$ and $a_2\in[n_3,n_4]$ so that $\mu_m(a_1)=\theta$ and $\mu_n(a_2)=\theta$. Now $b_1<a_1$ because $\bar{M}\approx\bar{P}$ and $b_2<a_2$ since $\bar{N}\approx\bar{Q}$. Hence, $b_1b_2<a_1a_2$. If $\bar{S}=\bar{M}\odot\bar{N}$ and $\bar{T}=\bar{P}\odot\bar{Q}$ then $\mu_s(a_1a_2)=\theta$ and $\mu_t(b_1b_2)=\theta$. Since $b_1b_2<a_1a_2$ the graph of $\mu_t(x)$ on $[p_1q_1,p_2q_2]$ must intersect the graph of $\mu_s(x)$ on $[m_3n_3,m_4n_4]$ at a y-coordinate above θ. Hence $\bar{M}\odot\bar{N}\approx\bar{P}\odot\bar{Q}$.

Lemma 3. *If $\bar{M}\approx\bar{P}$, then $\bar{M}^{1/s}\approx\bar{P}^{1/s}$, $s=2,3,\ldots$.*

Proof. Assume $\theta<1$ and $m_3<p_2$. Let the graph of $\mu_m(x)$ on $[m_3,m_4]$ intersect the graph of $\mu_p(x)$ on $[p_1,p_2]$ at $x=x_0$ and $\mu_m(x_0)=\mu_p(x_0)=\lambda\geq\theta$. Then the graph of the membership function for $\bar{M}^{1/s}$ on $[m_3^{1/s},m_4^{1/s}]$ intersects the graph of the membership function for $\bar{P}^{1/s}$ on $[p_1^{1/s},p_2^{1/s}]$ at $x_0^{1/s}$ with the common value $\lambda\geq\theta$. Therefore, $\bar{M}^{1/s}\approx\bar{P}^{1/s}$. The argument for $p_3<m_2$ is similar. If $[m_2,m_3]\cap[p_2,p_3]\neq\emptyset$, then it readily follows that $\bar{M}^{1/s}\approx\bar{P}^{1/s}$.

Now we return to the proof of Theorem 5. We show $\bar{a}_{il}^{k}\odot\bar{a}_{lj}^{k}\approx\bar{a}_{ij}^{k}$. First using Lemma 1 we may rewrite the product as

$$[(\bar{a}_{il}^{k1}\odot\bar{a}_{lj}^{k1})\odot\cdots\odot(\bar{a}_{il}^{kn}\odot\bar{a}_{lj}^{kn})]^{1/n}$$

because fuzzy multiplication is associative ([4], p. 45). Next we use Lemma 2 to obtain

$$(\bar{a}_{il}^{k1}\odot\bar{a}_{lj}^{k1})\odot\cdots\odot(\bar{a}_{il}^{kn}\odot\bar{a}_{lj}^{kn})\approx\bar{a}_{ij}^{k1}\odot\cdots\odot\bar{a}_{ij}^{kn}.$$

The result follows from Lemma 3.

6. Previous research

In a recent paper van Laarhoven and Pedrycz [13] extend Saaty's hierarchical analysis to fuzzy hierarchical analysis also using fuzzy numbers. They used Lootsma's [6] results on log least squares to extend Saaty's hierarchical analysis to the case of multiple estimates for the ratios and to the situation of missing data (no estimates for certain ratios).

For each ratio w_i^*/w_j^*, $i<j$, assume that we have n_{ij} estimates a_{ijk}, $k=1,2,\ldots,n_{ij}$, where some of the n_{ij} could be zero (missing data case). The log least squares estimate of the true weights w_i^* is a normalized solution $\hat{w}_i$ to

$$\min\left(\sum_{i<j}\sum_{k=1}^{n_{ij}}(\ln a_{ijk}-\ln(w_i/w_j))^2\right).$$

When $n_{ij}=1$, for $i<j$, the solution $\hat{w}_i$ is just the normalized geometric row mean discussed earlier. If $n_{ij}=n$, $i<j$, which is the multiple expert case discussed above, it can be shown that the solution $\hat{w}_i$ is the one indicated in Section 5. That is, one first obtains the geometric average across all the judges to obtain the 'average' positive reciprocal matrix and then computes the normalized geometric row means.

Van Laarhoven and Pedrycz assume the $\bar{a}_{ijk}$ are triangular fuzzy numbers and solve the normal equations for triangular fuzzy numbers $\bar{w}_i$ which are the fuzzy estimates of the true weights. Our method is to substitute the fuzzy ratio $\bar{a}_{ijk}$ into the solution of the normal equations. The solution to the normal equations is of the form

$$\hat{w}_i=F(a_{ijk}),$$

for some function F which depends on the n_{ij}. Our method is simply

$$\bar{w}_i=F(\bar{a}_{ijk}).$$

The paper by van Laarhoven and Pedrycz is subject to two main criticisms. First, there are situations where the normal equations do not have a unique solution for the triangular fuzzy numbers $\bar{w}_i$. We saw a similar situation when we attempted to employ Saaty's λ-max method to generate the fuzzy weights. Algebraic equations with fuzzy variables sometimes do not yield unique solutions. Secondly, they insist on obtaining triangular fuzzy numbers for their weights. Since algebraic operations on triangular fuzzy numbers do not necessarily produce a triangular fuzzy number, they are forced to employ approximate methods to preserve the shape of the fuzzy number.

Our method does not suffer from either of these problems and it always produces a unique fuzzy number for the weight. It is equally applicable to the cases of missing data and multiple estimates.

In a related paper [14] Wagenknecht and Hartmann also extend a method of estimating weights from a positive reciprocal matrix to fuzzy matrices. They are not attempting to develop fuzzy hierarchical analysis but instead they wish to use their results to choose a best solution from a set of efficient (undominated) solutions to a multicriteria decision problem. They consider two methods for determining the fuzzy weights. First they use least squares to estimate the best fuzzy weights $\bar{w}_i$ which approximate $\bar{a}_{ij}$ in the sense that

$$\bar{a}_{ij} \approx \bar{w}_i / \bar{w}_j.$$

Their fuzzy weights are (L, R)-fuzzy numbers [3; 4; 5, p. 53]. This procedure has two drawbacks: (1) they use an approximate expression for $\bar{w}_i/\bar{w}_j$; and (2) the least squares solution involves solving a nonlinear optimization problem with many variables. Their second method is similar to ours. The fuzzy weights are defined using the geometric mean but their fuzzy numbers $\bar{a}_{ij}$ are different. Since their fuzzy numbers are more complicated than ours they end up with a rather involved computation in order to obtain $\bar{w}_i$. We believe that our type of fuzzy number is more easily understood and used by experts (decision makers) and it is more easily manipulated mathematically to obtain the fuzzy weights.

7. Applications

In any application the value of θ, the number used in comparing fuzzy numbers, must be decided on first. There is no set rule which dictates the value of θ and we have suggested values like 0.7, 0.8, 0.9. A θ value of one implies that fuzzy number $\bar{M} = (m_1/m_2, m_3/m_4)$ is greater than another fuzzy number $\bar{N} = (n_1/n_2, n_3/n_4)$ if and only if $n_3 < m_2$. The ordering of fuzzy numbers is accomplished by comparing the intervals where their membership functions equal one when $\theta = 1$. In the following examples we have set $\theta = 0.8$. Notice that as θ increases from 0.7 to 0.8 to 0.9 fuzzy numbers become more spread out. That is, we may have $\bar{M} \approx \bar{N}$ for $\theta = 0.7$ but $\bar{M} > \bar{N}$ for $\theta = 0.8$.

Assume $\theta < 1$ and consider comparing two fuzzy numbers $\bar{M}$ and $\bar{N}$ when $n_3 < m_2$. Then $\bar{M} > \bar{N}$ if and only if

$$\max(\min(\mu_m(x), \mu_n(x))) < \theta,$$

where μ_m (μ_n) are the membership functions for $\bar{M}$ ($\bar{N}$). $\bar{M}$ is greater than $\bar{N}$ if the intersection of the graphs of μ_m and μ_n on $[n_3, m_2]$ lies below the horizontal line $y = \theta$. If this intersection lies on $y = \theta$ or above, then $\bar{M} \approx \bar{N}$. When $[n_2, n_3] \cap [m_2, m_3]$ is nonempty we have $\bar{M} \approx \bar{N}$ also.

Example 1. A government agency wishes to rank chemicals A_1, A_2, A_3 from most harmful to least harmful to the environment. The hierarchy is shown in Figure 1 with criterion C_1 = aquatic life, C_2 = agriculture and C_3 = timber. In a real study there would be more than three chemicals and possibly four or five criteria. The agency employs the testimony of a group of experts who supply the fuzzy positive reciprocal matrices $\bar{A}_{kl}$ for each criterion C_k. The agency also collects data on the pairwise comparisons of the criteria to obtain the fuzzy positive reciprocal matrices $\bar{E}_l$ for the criteria. Suppose the pooled information is given in Table 1. Each fuzzy positive reciprocal matrix is consistent.

Consider the fuzzy positive reciprocal matrix for criterion C_3 in Table 1. The fuzzy ratios have the following interpretations: (1) (1/2, 2/3) means that chemical

Table 1

Criterion		Fuzzy positive reciprocal matrix for criterion C_i in Example 1		
C_1		A_1	A_2	A_3
	A_1	1	(1/4/1/3, 1/3/1/2)	(1/2/1/2, 1/2/1/2)
	A_2	(2/3, 3/4)	1	(1/1, 2/2)
	A_3	(2/2, 2/2)	(1/2/1/2, 1/1)	1
C_2		A_1	A_2	A_3
	A_1	1	(6/6, 6/7)	(2/2, 4/4)
	A_2	(1/7/1/6, 1/6/1/6)	1	(1/2/1/2, 1/1)
	A_3	(1/4/1/4, 1/2/1/2)	(1/1, 2/2)	1
C_3		A_1	A_2	A_3
	A_1	1	(1/2, 2/3)	(7/8, 8/8)
	A_2	(1/3/1/2, 1/2/1)	1	(3/3, 4/4)
	A_3	(1/8/1/8, 1/8/1/7)	(1/4/1/4, 1/3/1/3)	1
		C_1	C_2	C_3
	C_1	1	(1/7/1/6, 1/6/1/5)	(1/3/1/2, 1/2/1)
	C_2	(5/6, 6/7)	1	(3/3, 3/3)
	C_3	(1/2, 2/3)	(1/3/1/3, 1/3/1/3)	1

A_1 is approximately twice as harmful to timber as chemical A_2; (2) (7/8, 8/8) means that A_1 is at most eight times as harmful as A_3; and (3) (3/3, 4/4) means A_2 is between 3 to 4 times as harmful as A_3.

It is not difficult to program a personal computer to determine the final fuzzy weight $\bar{f}_i$ and using graphics have their membership functions displayed on one coordinate system. The membership functions for the final fuzzy weights are

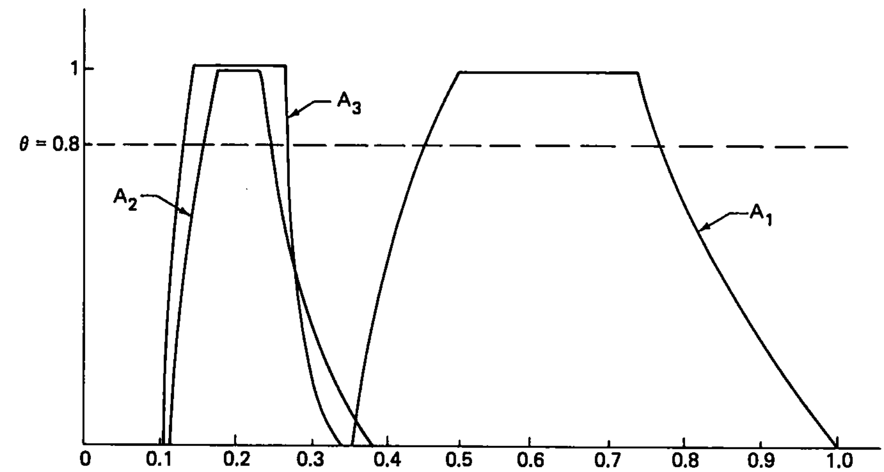

Fig. 2. Membership functions for the final fuzzy weights in Example 1.

shown in Figure 2. Therefore, chemical A_1 is the most harmful, and chemicals A_2 and A_3 are approximately equally harmful. That is, $H_1 = \{A_1\}$ and $H_2 = \{A_2, A_3\}$.

Example 2. A government wishes to rank various energy sources from most important to least important for the nation over the next ten years. The alternatives are A_1 = nuclear, A_2 = hydroelectric, A_3 = fossil, and A_4 = solar. The hierarchical structure is the same as in Figure 1. There are two criteria in the study. C_1 represents economical and political considerations including cost, balance of payments, etc. The other criterion C_2 represents military and defense considerations including self sufficiency, availability, etc. Expert opinion is solicited from energy experts, government officials, military personnel,... and the pooled data is presented in Table 2. Each fuzzy matrix is consistent.

Table 2

Criterion	Fuzzy positive reciprocal matrix for criterion C_i in Example 2				
C_1		A_1	A_2	A_3	A_4
	A_1	1	(1/7/1/6, 1/6/1/5)	(1/6/1/5, 1/5/1/4)	(1/1, 1/1)
	A_2	(5/6, 6/7)	1	(1/1, 2/2)	(4/4, 6/6)
	A_3	(4/5, 5/6)	(1/2/1/2, 1/1)	1	(3/4, 5/6)
	A_4	(1/1, 1/1)	(1/6/1/6, 1/4/1/4)	(1/6/1/5, 1/4/1/3)	1
C_2		A_1	A_2	A_3	A_4
	A_1	1	(1/5/1/5, 1/3/1/3)	(1/6/1/6, 1/6/1/5)	(1/2/1/2, 3/2/3/2)
	A_2	(3/3, 5/5)	1	(1/2/1/2, 1/1)	(6/6, 6/7)
	A_3	(5/6, 6/6)	(1/1, 2/2)	1	(8/9, 9/9)
	A_4	(2/3/2/3, 2/2)	(1/7/1/6, 1/6/1/6)	(1/9/1/9, 1/9/1/8)	1
		C_1	C_2		
	C_1	1	(1/2, 2/3)		
	C_2	(1/3/1/2, 1/2/1)	1		

The final fuzzy weights $\bar{f}_i$ for the alternatives A_i are shown in Figure 3. It is clear from Figure 3 that $H_1 = \{A_2, A_3\}$ and $H_2 = \{A_1, A_4\}$. This study has ranked hydroelectric and fossil fuel highest and approximately equally important. If it is desired to have H_1 contain only one alternative, then a second study comparing only A_2 and A_3 would be needed in order to try to differentiate between these two energy sources.

8. Summary and conclusions

This paper investigates the possibility of allowing participants in a hierarchical analysis to give vague, or imprecise, replys when comparing two alteratives. If a

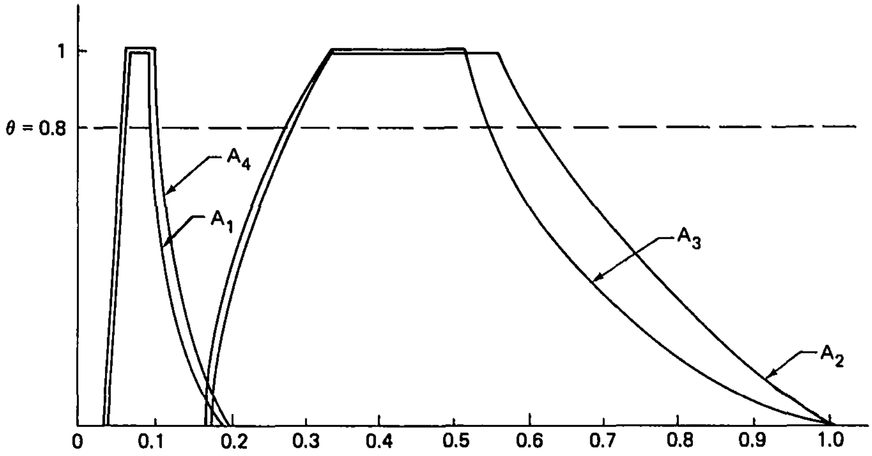

Fig. 3. Membership functions for the final fuzzy weights in Example 2.

person is comparing two alternatives A and B as to their relative importance (or weight, or brightness, etc.) and believes A is more important than B, then he/she may respond by saying A is approximately twice as important as B, or A is between 3 to 5 times as important as B, etc. Fuzzy numbers are used to capture the vagueness of these statements. Saaty's hierarchical analysis is generalized where fuzzy numbers are employed in place of exact ratios.

Much of classical hierarchical analysis may be accomplished with fuzzy numbers. This paper shows that the concept of consistency generalizes to fuzzy matrices. Saaty's λ-max method for determining the weights from a positive reciprocal matrix does not readily extend to fuzzy matrices. More research is needed on fuzzy eigenvalues and vectors of fuzzy positive reciprocal matrices. The geometric mean procedure is easily applied to a fuzzy matrix to obtain the fuzzy weights. The fuzzy weights are then combined in the usual way, depending on the hierarchical structure, to calculate the final fuzzy weights for the alternatives. The final fuzzy weights are utilized in partitioning the issues into sets $H_1, H_2, \ldots$. The set H_1 contains all the highest ranked alternatives, H_2 has those ranked second, etc. All the issues in H_1 are undominated and judged approximately equal. All the alternatives in H_k, $k > 1$, are approximately equal and are dominated by some issue in the next highest ranking H_{k-1}. Fuzzy hierarchical analysis is shown to possess a number of other desirable properties. The techniques readily extend to the situation where multiple experts are employed in the ranking process, or to the case of missing data.

If there are not too many alternatives and criteria fuzzy hierarchical analysis is easily programmed on a personal computer which could show graphically the membership functions for the final fuzzy weights. Then one could immediately pick off from the display of these membership functions the ranking $H_1, H_2, \ldots$.

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