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Theory and Methodology

Fuzzy hierarchical analysis revisited

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Abstract

We present a new method of finding the fuzzy weights in fuzzy hierarchical analysis which is the direct fuzzification of the original method used by Saaty in the analytic hierarchy process. We test our new procedure in two cases where there are formulas for the crisp weights. An example is presented where there are five criteria and three alternatives.

Keywords: Analytic hierarchy processes; Fuzzy sets

1. Introduction

In this section we first briefly review the contents of the paper, present our notation, and review the literature on fuzzy hierarchical analysis (FHA). In the next section we review the computational details of finding the weights in the original analytical hierarchical process. We fuzzify hierarchical analysis (HA) in the third section by allowing fuzzy numbers for the pairwise comparisons. Direct computation of fuzzy eigenvalues and fuzzy eigenvectors (the fuzzy weights) from a fuzzy, positive, reciprocal matrix is very complicated so instead we fuzzify an equivalent method to get the fuzzy weights. The new procedure is also quite involved so we design an evolutionary algorithm to estimate the fuzzy weights. To test our new method we compare the results to those obtained from 3×3 and 4×4 fuzzy, positive, reciprocal matrices, because for the 3×3 and 4×4 case there are formulas for the crisp weights. All this comprises the third section.

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The fourth section contains an example having five criteria and three alternatives. Also in this section we discuss consistency of fuzzy, positive, reciprocal matrices and how to obtain the final ranking based on fuzzy weights. The fifth section gives the results in the example and the final section contains our conclusions and suggestions for future research.

We place a "bar" over a letter to denote a fuzzy set. All our fuzzy sets will be fuzzy subsets of the real numbers. So, $\overline{a}_{ij}$, $\overline{w}_{ij}$, $\overline{b}$, $\overline{c}, \ldots$ are all fuzzy subsets of $\mathbb{R}$. If $\overline{a}$ is a fuzzy set, then $\overline{a}(x)$ is the value of the membership function at $x \in \mathbb{R}$. An α-cut of $\overline{a}$, written $\overline{a}[\alpha]$, is defined as $\{x|\overline{a}(x) \geqslant \alpha\}$ for $0 < \alpha \leqslant 1$. The support of $\overline{a}$, written $\overline{a}[0]$, is the closure of the union of $\overline{a}[\alpha]$, $0 < \alpha \leqslant 1$.

A trapezoidal fuzzy number $\overline{N}$ is defined by four numbers $\alpha \leqslant \beta \leqslant \gamma \leqslant \delta$. Assuming $\alpha < \beta < \gamma < \delta$, the graph of $y = \overline{N}(x)$ is a trapezoid with base on the interval $[\alpha, \delta]$ and $\overline{N}(x) = 1$ for $\beta \leqslant x \leqslant \gamma$. We get a triangular fuzzy number if $\beta = \gamma$. We write $\overline{N}$ as $(\alpha/\beta, \gamma/\delta)$. Special cases, when $\alpha = \beta$ or $\gamma = \delta$, etc. are all shown in Fig. 2.

In FHA one uses fuzzy numbers for the pairwise comparisons and the main problem is to compute the corresponding fuzzy weights. The direct approach, of finding fuzzy eigenvalues and fuzzy eigenvectors, is too computationally difficult [12,13,15,20], except for [24] to be discussed below, so researchers fuzzified another method. However, all these methods, except [24] and this paper, deviate from the original procedure used by Saaty in HA for finding the weights.

In [29] the authors, using the results in [30] on log least squares, extended HA to FHA. They used logarithmic regression to estimate the fuzzy weights (see also [31]). In their model they can have multiple estimates for each pairwise comparison and they can handle the problem of missing data (no estimates for certain comparisons). However, as pointed out in an example in [27] the logarithmic least square method can produce different weights, than Saaty's original procedure, for crisp data. In [11] the authors pointed out an error in [29] and they showed how to correct the procedure. However, in [25] it is shown that this method can produce fuzzy weights $\overline{w} = (w_1/w_2, w_2/w_3)$, triangular fuzzy numbers, with $w_3 < w_1$. That is, it is not a fuzzy number. This paper presents sufficient conditions so that you will get $w_1 < w_3$ for the triangular fuzzy number weights. This paper was followed by [26] where they define the concept of strong transitivity of a fuzzy, positive, reciprocal matrix (Section 3) and show that if this condition is satisfied, the log least squares method of Laarkoven and Pedrycz [29] produces triangular fuzzy weights with $w_1 < w_3$.

The logarithmic least squares method of obtaining fuzzy weights has been carried on in other papers. In [28] the authors present another solution to the problem using a generalized pseudoinverse approach but also points out you can get $w_3 < w_1$. The paper [37] uses "step-form" fuzzy numbers in logarithmic least squares to estimate these fuzzy weights, but they use a different objective function to be minimized in logarithmic regression.

There are also other papers in FHA using different procedures to compute fuzzy weights. In [36] they employed "step-form" fuzzy numbers and fuzzified another procedure, which they claim is the same as Saaty's original method for crisp perfectly consistent, positive, reciprocal matrices, to calculate the fuzzy weights. However, the matrices are usually not perfectly consistent, only "reasonably" consistent, so this procedure will produce different weights than Saaty's original method, for crisp data. The paper [34] uses fuzzy relational equations to model FHA problem. The modeling in [34] gives a fuzzy hierarchical process quite different from Saaty's original HA. The author in [44] develops a method for the interactive analysis of fuzzy pairwise comparisons in hierarchical weighting models which appears to us far removed from Saaty's original HA.

The series of papers [21–23,35] are also related to FHA. In [22,23,35], they changed a fuzzy, positive, reciprocal matrix into a crisp matrix, using α-cuts and convex combinations, and then computed the eigenvector (weight vector) from the crisp matrix. They do not obtain a fuzzy weight vector. Paper [21] is about speeding up the calculations in [35]. In our opinion, these papers are not about FHA since there are no fuzzy weights.

Paper [24] is in the spirit of Saaty's original HA. They first discuss a way of finding fuzzy $\lambda_{\max}$ (Section 2), where $\lambda_{\max}$ is the largest, positive, eigenvalue of a fuzzy, positive, reciprocal matrix. However, where they run into computational problems is in computing of the fuzzy eigenvector associated with $\lambda_{\max}$. We think it is a mistake to first compute fuzzy $\lambda_{\max}$ since it is not used in FHA. What is needed are the fuzzy weights.

In [12,20] the author also presents a method of computing the fuzzy weights in FHA. He used the fuzzification of the geometric mean of each row. If the positive, reciprocal matrix is perfectly consistent, then the geometric row mean procedure gives the same weights as the eigenvector method, which was Saaty's original method. However, we do not expect perfect consistency, so the geometric row procedure can give different weights than the eigenvector method.

In this paper we directly fuzzify Saaty's original method of computing the weights and for this reason we now believe we have the correct FHA. Since this paper follows [12,20] we have given it the title of FHA-Revisited.

Recently there have been a number of papers criticizing the methods used by Saaty in HA. See the papers [2–10,32,33,43,45], and the references in these papers, for a review of the literature. The criticisms include: (1) one should use the geometric means of the rows of a positive reciprocal matrix to calculate the weights and not the normalized eigenvector corresponding to $\lambda_{\max}$ (Eq. (2), Section 2); (2) Saaty's measure of consistency for a positive reciprocal matrix (Section 4.1) is incorrect; (3) the method of aggregating the weights across the criteria (Eq. (3), Section 2) is not correct; and (4) Saaty's procedure can produce rank reversals. The geometric mean method uses

$$w_i = \left(\prod_{j=1}^{n} a_{ij} \right)^{1/n}, \tag{1}$$

$1 \leqslant i \leqslant n$, to get the weights if $A = [a_{ij}]$ is a $n \times n$ positive reciprocal matrix. The papers [3–7,32,33] discuss reasons for using geometric mean instead of Saaty's procedure.

Acceptance of alternate methods in HA has been slow, and many researchers continue using the traditional HA methods as outlined in Section 2. In this paper we have adopted Saaty's original procedure but we have a companion paper [13,21] that bases FHA on the geometric mean method. So, this paper and [13,21] cover the two basic ways of calculating fuzzy weights in FHA. Also, in [45] the authors recommend fuzzifying the geometric mean procedure to obtain fuzzy weights in FHA.

2. Hierarchical analysis

In this section we review the basic computations needed to find the weights in HA. In HA a person (expert, judge) is asked to give ratios a_{ij} for each pairwise comparison between issues (alternatives, candidates) $A_1, \ldots, A_m$ for each criterion (objective) in a hierarchy, and also between the criteria. For some specific criterion C_k, if a person considers A_1 more important than A_5, then a_{15} might equal 3/1, or 5/1, or 7/1. The numbers for the ratios will be taken from the set $S = \{1, 2, 3, \ldots, 9\}$ so a_{15} could be s_1/s_5 with s_1, $s_5 \in S$ and $s_1 > s_5$. The ratios a_{ij} indicate, for this expert, the strength with which A_i dominates A_j. If $a_{15} = 5/1$ then $a_{51} = 1/5$. That is, $a_{ij} = (a_{ji})^{-1}$, all i, j, with $a_{ii} = 1$, $1 \leqslant i \leqslant m$. Let $\mathscr{A}$ be the $m \times m$ matrix whose entries are the ratios ($a_{ji} = a_{ij}^{-1}$). $\mathscr{A}$ is called a positive reciprocal matrix. Since $\mathscr{A}$ is for criterion C_k we will now write $\mathscr{A}_k$ for this matrix.

Assume there are K criteria $C_1, \ldots, C_K$ with a positive reciprocal matrix $\mathscr{A}_k$ for each C_k, $1 \leqslant k \leqslant K$. Also, the judge must give pairwise comparisons of the criteria producing a positive reciprocal matrix $\mathscr{E}$. This hierarchical structure is shown in Fig. 1. Examples, with actual fuzzy numbers in the $\mathscr{A}_k$ and $\mathscr{E}$, are presented in Sections 3 and 4.

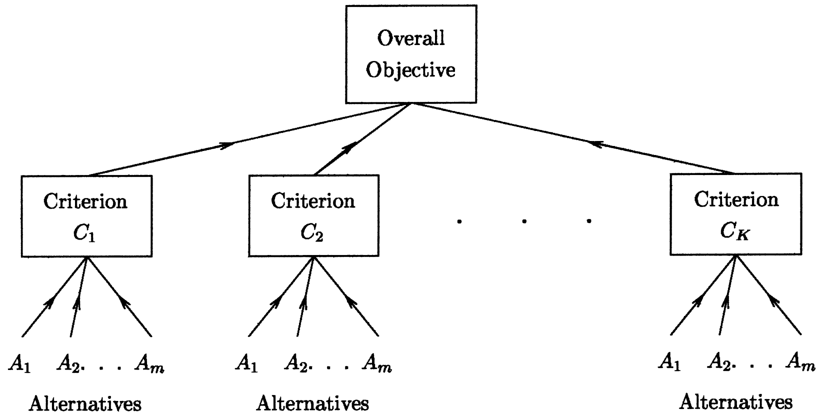

Fig. 1. Hierarchical structure.

Next one computes weights $w_k^{\mathrm{T}} = (w_{1k}, \ldots, w_{mk})$ for each $\mathscr{A}_k$ and $e^{\mathrm{T}} = (e_1, \ldots, e_K)$ for $\mathscr{E}$. Given any positive reciprocal matrix $\mathscr{A}$, let the eigenvalues, counting a root of multiplicity n n-times, be $\lambda_1, \ldots, \lambda_m$. There is a dominant (real, positive) eigenvalue, let us call it $\lambda_{\max}$, so that $|\lambda_i| < \lambda_{\max}$ for all $\lambda_i \neq \lambda_{\max}$. Also, $\lambda_{\max}$ is a root of multiplicity 1. Corresponding to $\lambda_{\max}$ there is a unique eigenvector $w^{\mathrm{T}} = (w_1, \ldots, w_m)$ so that

$$\mathscr{A}w = \lambda_{\max}w, \tag{2}$$

where $w_i > 0$ for all i and $\sum_{i=1}^{m} w_i = 1$. This positive, normalized (sum one), vector w gives the weights for $\mathscr{A}$ [38–42]. Then, w_k is the positive, normalized, eigenvector corresponding to $\lambda_{\max}$ for $\mathscr{A}_k$, $1 \leqslant k \leqslant K$, and e is the eigenvector for $\mathscr{E}$.

The objective of HA is to rank the alternatives across all the criteria. Then, assuming that the reciprocal matrices $\mathscr{A}_k$, $1 \leqslant k \leqslant K$, and $\mathscr{E}$ are reasonably consistent [38–42], the final ranking of the alternatives is determined by the vector $r^{\mathrm{T}} = (r_1, \ldots, r_m)$ where

$$r_j = \sum_{k=1}^{K} w_{jk} e_k, \tag{3}$$

$1 \leqslant j \leqslant m$. We will discuss consistency for fuzzy hierarchical analysis in Section 4. The weight for alternative A_j is r_j, $1 \leqslant j \leqslant m$. The alternatives are ranked according to the numbers r_j, $1 \leqslant j \leqslant m$. The hierarchical structure (Fig. 1) can be expanded to more levels but we shall consider, in this paper, only the three levels shown in Fig. 1.

The procedure described above does not easily extend to FHA discussed in the next section. In FHA we have fuzzy numbers in the $\mathscr{A}_k$ and $\mathscr{E}$. For each $\mathscr{A}_k$ and $\mathscr{E}$ we would need to find $\lambda_{\max}$, now a fuzzy number, and the corresponding (positive, normalized) vector of fuzzy numbers. The computations here are quite involved [12,13,15,20], so instead we will employ another computational method of obtaining the weights. This procedure will be extended to FHA in the next section.

Let $1^{\mathrm{T}} = (1, 1, \ldots, 1)$, a vector of length m of all ones, let $\mathscr{A}$ be any positive reciprocal matrix with sum(l) =sum of all the elements in $\mathscr{A}^l$, $l = 1, 2, 3, \ldots$ Define

$$\lim_{l \to \infty} \left(\frac{\mathscr{A}^l \cdot 1}{\mathrm{sum}(l)} \right) = z, \tag{4}$$

then if

$$w = \left(\sum_{i=1}^{m} z_i\right)^{-1} z, \tag{5}$$

we know that [38,40–42] w is the unique, positive, normalized eigenvector of $\mathscr{A}$ corresponding to $\lambda_{\max}$. This gives us a way of computing the weights w_k for $\mathscr{A}_k$ and e of $\mathscr{E}$. We evaluate Eq. (4) for $l = 2$, $l = 4$ (square), $l = 8$ (square again), ... until the vector stabilizes (changes from step-to-step are less than a given $\varepsilon > 0$). Then, Eq. (5) produces the approximation to w.

3. Fuzzy hierarchical analysis

The experts are allowed to use fuzzy ratios in place of exact ratios. The $\overline{a}_{ij}$, $i \neq j$, can now be fuzzy numbers in any positive reciprocal matrix. As before, $\overline{a}_{ii} = 1$ all i. The types of fuzzy numbers that can be used in paired comparisons are described by $\overline{a}_{ij} = (\alpha/\beta, \gamma/\delta)$ where α, β, γ, $\delta \in S$, $\alpha \leqslant \beta \leqslant \gamma \leqslant \delta$. There are eight types of fuzzy numbers, with a real number represented by $\alpha = \beta = \gamma = \delta$, and the other seven are shown in Fig. 2. The judge can input these fuzzy numbers various ways including drawing them or using their verbal equivalents. For the trapezoidal (Fig. 2(a)) on could say approximately between β to 1 and γ to 1. The triangle (Fig. 2(b)) is approximately β to 1. If $\overline{a}_{ij} = (\alpha/\beta, \gamma/\delta)$, then $\overline{a}_{ij}^{-1} = \overline{a}_{ji} = (\delta^{-1}/\gamma^{-1}, \beta^{-1}/\alpha^{-1})$, the reciprocal of the fuzzy number $\overline{a}_{ij}$. The reciprocals are shown in Fig. 3.

The application domain of this paper is HA, according to Saaty, where the experts (judges) are allowed to express uncertainty, using the fuzzy sets in Fig. 2, in their pairwise comparisons.

We will need α-cuts of all these fuzzy numbers in Figs. 2 and 3. It is obvious on how to take α-cuts of the fuzzy numbers in Figs. 2(a),(b) and 3(a),(b). The procedure is simple for Figs. 2(c) and (d) so consider Fig. 2(c). The α-cut is $[\alpha, \delta + (\gamma - \delta)\alpha]$, $0 \leqslant \alpha \leqslant 1$. So, an α-cut of the fuzzy number in Fig. 3(c) is $[(\delta + (\gamma - \delta)\alpha)^{-1}, \alpha^{-1}]$, $0 \leqslant \alpha \leqslant 1$. For Fig. 2(e) all α-cuts are $[\alpha, \delta]$. In Fig. 2(g), an α-cut is $[\alpha + (\beta - \alpha)\alpha, \delta]$ and its reciprocal gives an α-cut for Fig. 3(g).

Now we assume the elements in the fuzzy positive reciprocal matrices $\overline{\mathscr{A}}_k$ and $\overline{\mathscr{E}}$ are $\overline{a}_{ij} = (\alpha/\beta, \gamma/\delta)$, $\overline{a}_{ii} = 1$ and $\overline{a}_{ji} = \overline{a}_{ij}^{-1}$. Some of the $\overline{a}_{ij}$ can be real numbers $\overline{a}_{ij} = (\alpha/\alpha, \alpha/\alpha)$. We now describe how we are going to compute the fuzzy weight vectors $\overline{w}_k$ and $\overline{e}$. There are a number of other issues to be addressed in FHA, like consistency, and how do we obtain the final ranking because now the weight r_j (Eq. (3)) for alternative A_j will be a fuzzy number. These two issues will be considered in Section 4. Right now we are only concerned with finding the fuzzy weight vector for a fuzzy, positive, reciprocal matrix.

We compute the fuzzy weight vector by fuzzifying Eqs. (4) and (5). Let $\overline{\mathscr{A}}$ be a fuzzy positive, reciprocal matrix. Choose $\alpha \in [0,1]$. Let $\Gamma(\alpha) = \prod\{\overline{a}_{ij}[\alpha] \mid 1 \leqslant i < j \leqslant m\}$ and $v \in \Gamma(\alpha)$ we write as $v = (a_{12}, \ldots, a_{1m}, a_{23}, \ldots, a_{m-1,m})$. Define positive, reciprocal, matrix $\mathscr{A} = [e_{ij}]$ as follows: (1) $e_{ij} = a_{ij}$ if $1 \leqslant i < j \leqslant m$; (2) $e_{ii} = 1$, $1 \leqslant i \leqslant m$; and (3) $e_{ji} = a_{ij}^{-1}$ for $1 \leqslant i < j \leqslant m$. Let

$$z = \lim_{l \to \infty} \left(\frac{\mathscr{A}^l \cdot 1}{\mathrm{sum}(l)} \right), \tag{6}$$

and define $w_v = (\sum_{i=1}^{m} z_i)^{-1} z$. Set $w_v^{\mathrm{T}} = (w_{v1}, \ldots, w_{vm})$. We have described a continuous mapping $\Phi_i(v) = w_{vi}$, $1 \leqslant i \leqslant m$, for each α in $[0,1]$. So, let [17]:

$$w_{i1}(\alpha) = \min\{w_{vi} \mid v \in \Gamma(\alpha)\}, \tag{7}$$

$$w_{i2}(\alpha) = \max\{w_{vi} \mid v \in \Gamma(\alpha)\}, \tag{8}$$

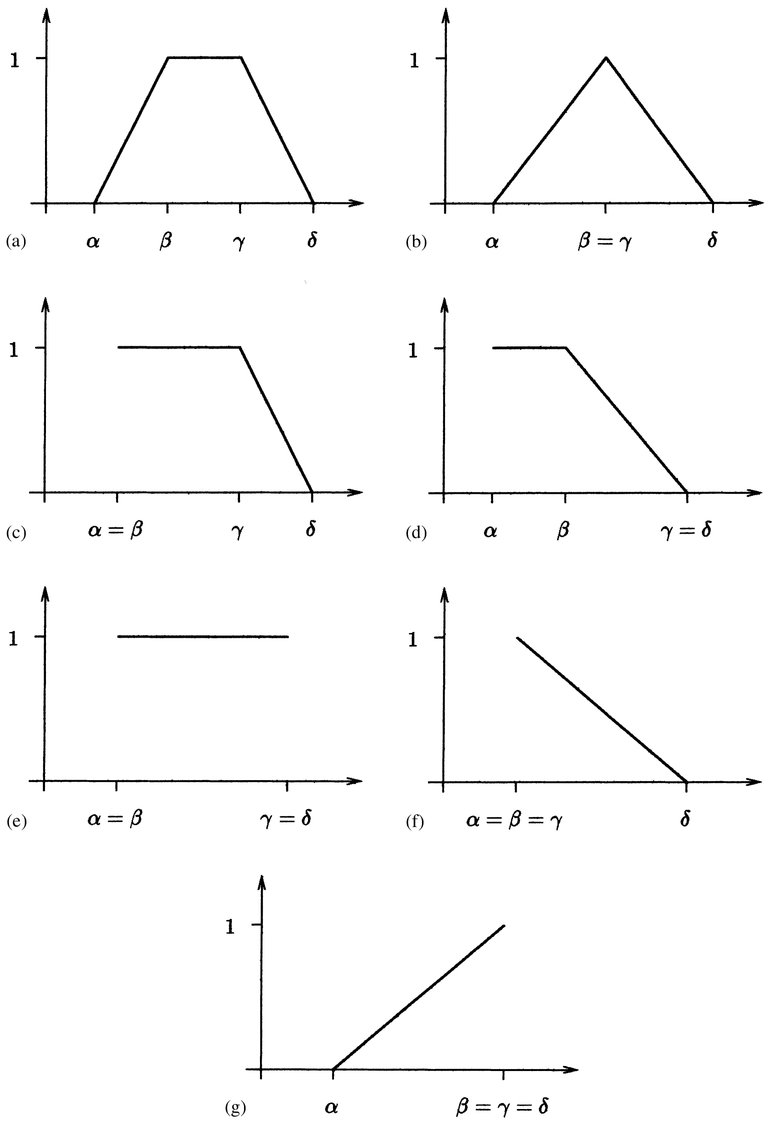

Fig. 2. Fuzzy numbers in FHA: (a) trapezoidal, (b) triangle, (c) more than α to 1, (d) less than δ to 1, (e) between $\alpha/1$ and $\gamma/1$, (f) at least $\alpha/1$, and (g) at most $\delta/1$.

$1 \leqslant i \leqslant m$, for all $\alpha \in [0,1]$. Then, $[w_{i1}(\alpha), w_{i2}(\alpha)]$ becomes α-cuts of fuzzy numbers $\overline{w}_i$, $1 \leqslant i \leqslant m$, which produce the fuzzy weight vector $\overline{w}^{\mathrm{T}} = (\overline{w}_1, \ldots, \overline{w}_m)$.

It is no easy job to compute the $w_{i1}(\alpha)$ and $w_{i2}(\alpha)$, so we propose an evolutionary algorithm (EA) to do this job. The basic EA is described in Appendix A. Let us here briefly describe how the EA works to find $\overline{w}$.

Computing $w_{i1}(\alpha)$ is a complicated, non-linear, optimization problem and EAs are very good search tools for optimization. The search space is $\Gamma(\alpha)$, so members of the population will be vectors v in $\Gamma(\alpha)$. We will estimate $w_{i1}(\alpha)$ $(w_{i2}(\alpha))$ for selected values of α, say $\alpha = 0$, 0.2, 0.4, 0.6, 0.8 and 1. Then, for each

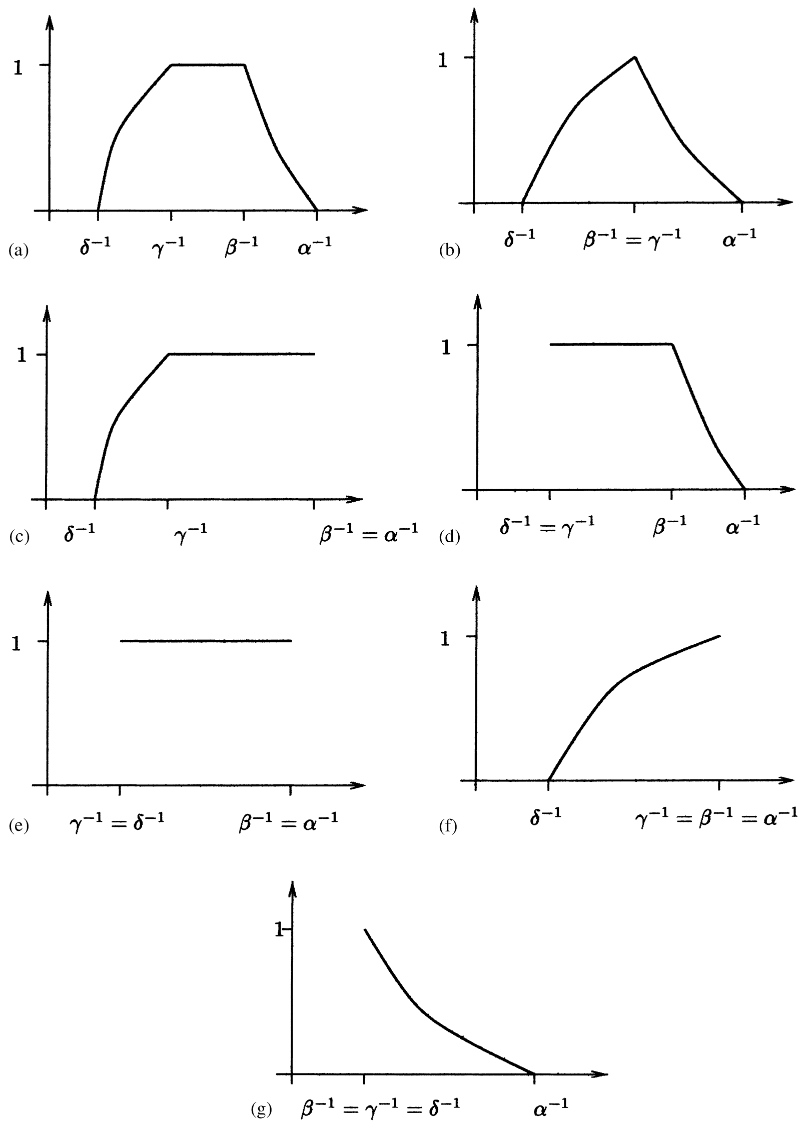

Fig. 3. Reciprocals of fuzzy numbers in Fig. 2: (a) reciprocal of Fig. 2(a), (b) reciprocal of Fig. 2(b), (c) reciprocal of Fig. 2(c), (d) reciprocal of Fig. 2(d), (e) reciprocal of Fig. 2(e), (f) reciprocal of Fig. 2(f), and (g) reciprocal of Fig. 2(g).

$i = 1, 2, \ldots, m$ and each $\alpha = 0, 0.2, \ldots, 1$ we run the EA to approximate the minimum of $\Phi_i(v) = w_{i1}(\alpha)$. Then, for each $i = 1, 2, \ldots, m$ and each $\alpha = 0.0, 0.2, \ldots, 1.0$ we apply the EA to approximate the maximum of $\Phi_i(v) = w_{i2}(\alpha)$. This produces approximations to $[w_{i1}(\alpha), w_{i2}(\alpha)] = \overline{w}_i[\alpha]$ and the fuzzy weight vector $\overline{w}$ for $\overline{\mathscr{A}}$.

What we have done is fuzzify the original method of Saaty (Eqs. (4) and (5)) for finding the weights to produce the fuzzy weights. We have used this procedure many times before [14–16,18,19] with success. The general procedure is: (1) find the crisp solution; (2) fuzzify the crisp solution; and (3) the fuzzified crisp solution gives the answer to the fuzzy problem.

In order to test our procedure we will consider two special cases of $m = 3$ and $m = 4$. When $m = 3$ and $m = 4$ there are formulas for the unique positive, normalized, eigenvector corresponding to $\lambda_{\max}$.

3.1. $m = 3$

Let

$$\mathscr{A} = \begin{bmatrix} 1 & a & b \\ a^{-1} & 1 & c \\ b^{-1} & c^{-1} & 1 \end{bmatrix} \tag{9}$$

be the positive, reciprocal, matrix. If $w^{\mathrm{T}} = (w_1, w_2, w_3)$ is the unique positive, normalized, eigenvector corresponding to $\lambda_{\max}$, then we know that [40]:

$$w_1 = \frac{a^{1/3}b^{1/3}}{T} = f_1(a,b,c), \tag{10}$$

$$w_2 = \frac{a^{-1/3}c^{1/3}}{T} = f_2(a,b,c), \tag{11}$$

$$w_3 = \frac{b^{-1/3}c^{-1/3}}{T} = f_3(a,b,c), \tag{12}$$

where

$$T = a^{1/3}b^{1/3} + a^{-1/3}c^{1/3} + b^{-1/3}c^{-1/3}. \tag{13}$$

We now fuzzify w_1, w_2, w_3 by substituting fuzzy numbers $\overline{a}$ for a, $\overline{b}$ for b, and $\overline{c}$ for c, and use the extension principle to find fuzzy weights $\overline{w}_1$, $\overline{w}_2$ and $\overline{w}_3$.

Let us look more closely on how we are to get $\overline{w}_1$, $\overline{w}_2$, $\overline{w}_3$. Since the f_i are continuous we know that [17]:

$$w_{i1}(\alpha) = \min\left\{f_i(a,b,c) \mid a \in \overline{a}[\alpha],\ b \in \overline{b}[\alpha], c \in \overline{c}[\alpha]\right\}, \tag{14}$$

$$w_{i2}(\alpha) = \max\left\{f_i(a,b,c) \mid a \in \overline{a}[\alpha],\ b \in \overline{b}[\alpha], c \in \overline{c}[\alpha]\right\} \tag{15}$$

for $i = 1,2,3$, $\alpha \in [0,1]$, where $[w_{i1}(\alpha), w_{i2}(\alpha)]$ is an α-cut of $\overline{w}_i$. It is easy to see that: (1) $\partial f_1/\partial a > 0$, $\partial f_1/\partial b > 0$; (2) $\partial f_2/\partial a < 0$, $\partial f_2/\partial c > 0$; and (3) $\partial f_3/\partial b < 0$, $\partial f_3/\partial c < 0$. However, f_1 may be increasing for some c and decreasing for other c. Similarly, for f_2 a function of b and f_3 a function of a. Hence

$$w_{11}(\alpha) = f_1(a_1(\alpha), b_1(\alpha), c^*), \tag{16}$$

$$w_{12}(\alpha) = f_1(a_2(\alpha), b_2(\alpha), c^{**}) \tag{17}$$

for c^*, c^{**} in $\overline{c}[\alpha]$, and

$$w_{21}(\alpha) = f_2(a_2(\alpha), b^*, c_1(\alpha)), \tag{18}$$

$$w_{22}(\alpha) = f_2(a_1(\alpha), b^{**}, c_2(\alpha)) \tag{19}$$

for b^*, b^{**} in $\overline{b}[\alpha]$, and

$$w_{31}(\alpha) = f_3(a^*, b_2(\alpha), c_2(\alpha)), \tag{20}$$

$$w_{32}(\alpha) = f_3(a^{**}, b_1(\alpha), c_1(\alpha)) \tag{21}$$

for a^*, a^{**} in $\overline{a}[\alpha]$, where $\overline{a}[\alpha] = [a_1(\alpha), a_2(\alpha)]$, $\overline{b}[\alpha] = [b_1(\alpha), b_2(\alpha)]$, and $\overline{c}[\alpha] = [c_1(\alpha), c_2(\alpha)]$. In this way we find the $\overline{w}_i$, $1 \leqslant i \leqslant 3$.

Let us call the procedure of calculating the weights using the expressions in this section, Method II. Method I will use the EA and Eqs. (7) and (8).

To test Method I we used both methods on the following positive reciprocal matrix:

$$\mathscr{A} = \begin{bmatrix} 1 & a & b \\ a^{-1} & 1 & 3 \\ b^{-1} & 1/3 & 1 \end{bmatrix} \tag{22}$$

from [12,20], for $a = (5/6, 6/7)$, $b = (1/2, 2/3)$. The results are shown in Table 1. We see that the EA computes a good approximation to the fuzzy weights obtained using Method II.

3.2. $m = 4$

Let

$$\mathscr{A} = \begin{bmatrix} 1 & a & b & c \\ a^{-1} & 1 & d & e \\ b^{-1} & d^{-1} & 1 & f \\ c^{-1} & e^{-1} & f^{-1} & 1 \end{bmatrix} \tag{23}$$

Table 1
Testing the evolutionary algorithm method (Method I) on a 3 × 3 example problem

α	Method I	Method II
	$\overline{w}_1[\alpha]$	$\overline{w}_1[\alpha]$
0	[0.5273, 0.6893]	[0.5267, 0.6908]
0.2	[0.5538, 0.6793]	[0.5535, 0.6806]
0.4	[0.5771, 0.6664]	[0.5765, 0.6695]
0.6	[0.5970, 0.6561]	[0.5965, 0.6575]
0.8	[0.6153, 0.6442]	[0.6142, 0.6445]
1.0	[0.6301, 0.6301]	[0.6301, 0.6301]
α	$\overline{w}_2[\alpha]$	$\overline{w}_2[\alpha]$
0	[0.1893, 0.2590]	[0.1888, 0.2598]
0.2	[0.1951, 0.2497]	[0.1940, 0.2503]
0.4	[0.2005, 0.2408]	[0.1996, 0.2415]
0.6	[0.2076, 0.2330]	[0.2055, 0.2332]
0.8	[0.2119, 0.2254]	[0.2118, 0.2256]
1.0	[0.2184, 0.2184]	[0.2184, 0.2184]
α	$\overline{w}_3[\alpha]$	$\overline{w}_3[\alpha]$
	[0.1206, 0.2131]	[0.1204, 0.2136]
0.2	[0.1259, 0.1958]	[0.1254, 0.1962]
0.4	[0.1313, 0.1820]	[0.1309, 0.1821]
0.6	[0.1374, 0.1698]	[0.1370, 0.1703]
0.8	[0.1442, 0.1599]	[0.1438, 0.1602]
1.0	[0.1515, 0.1515]	[0.1515, 0.1515]

be a positive, reciprocal matrix. If $w^{\mathrm{T}} = (w_1, w_2, w_3, w_4)$ is the positive, normalized, eigenvector of $\mathscr{A}$ corresponding to $\lambda_{\max}$, then there are formulas [1] [40] $w_i = f_i(a, b, c, d, e, f)$, $1 \leqslant i \leqslant 4$. We fuzzify these expressions producing the fuzzy weights $\overline{w}_i = f_i(\overline{a}, \overline{b}, \ldots, \overline{f})$, $1 \leqslant i \leqslant 4$. As before, we use the extension principle to find $\overline{w}_i$, for all i, so that [17]:

$$w_{i1}(\alpha) = \min\{f_i(a, b, c, d, e, f) \mid a \in \overline{a}[\alpha], \ldots, f \in \overline{f}[\alpha]\}, \tag{24}$$

$$w_{i2}(\alpha) = \max\{f_i(a, b, c, d, e, f) \mid a \in \overline{a}[\alpha], \ldots, f \in \overline{f}[\alpha]\} \tag{25}$$

for $1 \leqslant i \leqslant 4$, $\alpha \in [0, 1]$. The expressions f_i are now quite complicated so it is very difficult to compute the $w_{ij}(\alpha)$, $1 \leqslant i \leqslant 4$, $1 \leqslant j \leqslant 2$, $\alpha \in [0, 1]$. So, we wrote another EA to estimate the $w_{ij}(\alpha)$ $1 \leqslant i \leqslant 4$, $1 \leqslant j \leqslant 2$, $\alpha = 0, 0.2, \ldots, 1.0$ in Eqs. (24) and (25).

We tested Method I again on the following 4×4 matrix:

$$\mathscr{A} = \begin{bmatrix} 1 & a & b & 1 \\ a^{-1} & 1 & d & e \\ b^{-1} & d^{-1} & 1 & f \\ 1 & e^{-1} & f^{-1} & 1 \end{bmatrix}, \tag{26}$$

where $a = (5/6, 6/7)$, $b = (4/5, 5/6)$, $d = (1/1, 2/2)$, $e = (4/4, 6/6)$, $f = (3/4, 5/6)$. The positive reciprocal matrix was also used in [12,20]. The results are displayed in Table 2. Again, Method I is a good approximation to the fuzzy weights $\overline{w}_i$, $1 \leqslant i \leqslant 4$, assuming that Method II also produced good estimates. We conclude that our new Method I can obtain good estimates of the fuzzy weights.

4. Application

This application has been developed from an example in [38,39]. A recent college graduate has be offered three jobs A_1, A_2, A_3. In order to rank these jobs he evaluates each with respect to five criteria: (1) C_1 = pay; (2) C_2 = benefits; (3) C_3 = location; (4) C_4 = colleagues (fellow workers); and (5) C_5 = potential for advancement. Using FHA he constructs the following fuzzy reciprocal matrices:

$$\overline{\mathscr{A}}_1 = \begin{bmatrix} 1 & (3/3, 5/5)^{-1} & 1/2 \\ (3/3, 5/5) & 1 & (2/3, 3/4) \\ 2 & (2/3, 3/4)^{-1} & 1 \end{bmatrix} \tag{27}$$

for C_1 = pay,

$$\overline{\mathscr{A}}_2 = \begin{bmatrix} 1 & (2/3, 3/4)^{-1} & (2/3, 3/4)^{-1} \\ (2/3, 3/4) & 1 & 1 \\ (2/3, 3/4) & 1 & 1 \end{bmatrix} \tag{28}$$

for C_2 = benefits,

$$\overline{\mathscr{A}}_3 = \begin{bmatrix} 1 & 1 & (7/7, 8/10) \\ 1 & 1 & (7/8, 9/10) \\ (7/7, 8/10)^{-1} & (7/8, 9/10)^{-1} & 1 \end{bmatrix} \tag{29}$$

[1] One must first correct the expression for $\lambda_{\max}$.

Table 2
Testing Method I on a 4×4 example problem

α	Method I	Method II
	$\overline{w}_1[\alpha]$	$\overline{w}_1[\alpha]$
0	$[0.4521, 0.5283]$	$[0.4523, 0.5291]$
0.2	$[0.4602, 0.5219]$	$[0.4594, 0.5225]$
0.4	$[0.4672, 0.5157]$	$[0.4667, 0.5165]$
0.6	$[0.4726, 0.5108]$	$[0.4739, 0.5113]$
0.8	$[0.4819, 0.5043]$	$[0.4816, 0.5054]$
1.0	$[0.4891, 0.5012]$	$[0.4887, 0.4999]$
α	$\overline{w}_2[\alpha]$	$\overline{w}_2[\alpha]$
0	$[0.1667, 0.2579]$	$[0.1616, 0.2589]$
0.2	$[0.1671, 0.2533]$	$[0.1655, 0.2543]$
0.4	$[0.1686, 0.2510]$	$[0.1693, 0.2492]$
0.6	$[0.1724, 0.2473]$	$[0.1729, 0.2467]$
0.8	$[0.1765, 0.2443]$	$[0.1761, 0.2416]$
1.0	$[0.1792, 0.2383]$	$[0.1788, 0.2375]$
α	$\overline{w}_3[\alpha]$	$\overline{w}_3[\alpha]$
0	$[0.1251, 0.2303]$	$[0.1247, 0.2309]$
0.2	$[0.1290, 0.2251]$	$[0.1295, 0.2254]$
0.4	$[0.1352, 0.2201]$	$[0.1346, 0.2209]$
0.6	$[0.1412, 0.2128]$	$[0.1400, 0.2132]$
0.8	$[0.1463, 0.2102]$	$[0.1454, 0.2097]$
1.0	$[0.1504, 0.2041]$	$[0.1511, 0.2045]$
α	$\overline{w}_4[\alpha]$	$\overline{w}_4[\alpha]$
0	$[0.1123, 0.1433]$	$[0.1120, 0.1431]$
0.2	$[0.1130, 0.1424]$	$[0.1131, 0.1416]$
0.4	$[0.1138, 0.1410]$	$[0.1141, 0.1401]$
0.6	$[0.1153, 0.1400]$	$[0.1151, 0.1388]$
0.8	$[0.1159, 0.1389]$	$[0.1161, 0.1370]$
1.0	$[0.1162, 0.1361]$	$[0.1169, 0.1353]$

for $C_3 =$ location,

$$\overline{\mathscr{A}}_4 = \begin{bmatrix} 1 & (1/3, 3/3)^{-1} & (2/2, 2/5) \\ (1/3, 3/3) & 1 & (6/7, 7/8) \\ (2/2, 2/5)^{-1} & (6/7, 7/8)^{-1} & 1 \end{bmatrix} \tag{30}$$

for $C_4 =$ colleagues,

$$\overline{\mathscr{A}}_5 = \begin{bmatrix} 1 & (4/4, 4/6)^{-1} & (3/4, 5/5)^{-1} \\ (4/4, 4/6) & 1 & 1 \\ (3/4, 5/5) & 1 & 1 \end{bmatrix} \tag{31}$$

for $C_5 =$ advancement, and

$$\overline{\mathscr{E}} = \begin{array}{c} \\ P \\ B \\ L \\ Co \\ Av \end{array} \begin{array}{c} \begin{array}{ccccc} P & B & L & Co & Av \end{array} \\ \begin{pmatrix} 1 & (1/2,2/3) & (3/3,5/5)^{-1} & 1 & (4/4,6/6)^{-1} \\ (1/2,2/3)^{-1} & 1 & 1/6 & (1/2,4/5)^{-1} & 1/8 \\ (3/3,5/5) & 6 & 1 & 3 & 1 \\ 1 & (1/2,4/5) & 1/3 & 1 & 1/4 \\ (4/4,6/6) & 8 & 1 & 4 & 1 \end{pmatrix} \end{array}, \tag{32}$$

for the criteria, where $P=$ pay, $B=$ benefits, $L=$ location, $Co=$ colleagues, and $Av=$ advancement. In the $\overline{\mathscr{A}}_i$ matrices the: (1) first row/column corresponds to alternative A_1; (2) second row/colum is A_2; and (3) the third row/column is for job A_3. Using our EA we compute the fuzzy weight vectors $\overline{w}_k$ for $\overline{\mathscr{A}}_k$, $1 \leqslant k \leqslant 5$, and $\overline{e}$ for $\overline{\mathscr{E}}$. Then, from Eq. (3), we get

$$\overline{r}_j = \sum_{k=1}^{5} \overline{w}_{jk}\overline{e}_k \tag{33}$$

for all j. The fuzzy weight for job A_j is $\overline{r}_j$. However, before showing the results we need to discuss consistency and the ranking of fuzzy numbers.

4.1. Consistency

Let $\mathscr{A}$ be a positive, reciprocal, matrix. $\mathscr{A}$ is said to be consistent when $a_{ik}a_{kj} = a_{ij}$ for all i, j, k. This means that if the judge states $a_{ik} = 2/1$ for A_i versus A_k and gives $a_{kj} = 3/1$ for A_k against A_j, then to be logically consistent this judge should state $6/1$ for A_i versus A_j. If $\mathscr{A}$ is consistent then $\lambda_{\max} = m$ and in general $\lambda_{\max} \geqslant m$. So a measure of consistency is built around the difference $(\lambda_{\max} - m)$ (see [38,40–42]). We would say $\mathscr{A}$ is "reasonably" consistent when $(\lambda_{\max} - m)$ is not too large (maybe $\lambda_{\max} - m \leqslant 1$).

To talk about consistency for fuzzy, positive, reciprocal matrices we first need to define what is meant by $\overline{M} \geqslant \overline{N}$, $\overline{M} > \overline{N}$ and $\overline{M} \approx \overline{N}$ for two fuzzy numbers $\overline{M}$ and $\overline{N}$. Define (see [12,20])

$$v(\overline{M} \geqslant \overline{N}) = \sup_{x \geqslant y}(\min(\overline{M}(x), \overline{N}(y)). \tag{34}$$

We then write $\overline{M} > \overline{N}$ if $v(\overline{M} \geqslant \overline{N}) = 1$ and $v(\overline{N} \geqslant \overline{M}) < \theta$, where θ is some fixed positive fraction less than 1. Let us use $\theta = 0.8$ in this paper. Next, we write $\overline{M} \approx \overline{N}$ when $\overline{M}$ is not greater than $\overline{N}$ and $\overline{N}$ is not greater than $\overline{M}$. Or, if

$$\min(v(\overline{M} \geqslant \overline{N}),\ v(\overline{N} \geqslant \overline{M})) \geqslant \theta, \tag{35}$$

then $\overline{M} \approx \overline{N}$. Finally, we say $\overline{M} \geqslant \overline{N}$ if $\overline{M} > \overline{N}$ or $\overline{M} \approx \overline{N}$.

A fuzzy, positive, reciprocal matrix $\overline{\mathscr{A}} = [\overline{a}_{ij}]$ is defined to be consistent when

$$\overline{a}_{ik} \cdot \overline{a}_{kj} \approx \overline{a}_{ij} \tag{36}$$

for all i, j, k. The following theorem was proven in [12].

Theorem 1. *Let* $\overline{\mathscr{A}} = [\overline{a}_{ij}]$ *be a fuzzy, positive, reciprocal matrix with* $\overline{a}_{ij} = (\alpha_{ij}/\beta_{ij}, \gamma_{ij}/\delta_{ij})$. *Choose* $a_{ij} \in [\beta_{ij}, \gamma_{ij}]$ *and form* $\mathscr{A} = [a_{ij}]$. *If* $\mathscr{A}$ *is consistent, then* $\overline{\mathscr{A}}$ *is consistent.*

We shall not demand all $\overline{\mathscr{A}}_k$ and $\overline{\mathscr{E}}$ to be perfectly consistent. All we shall ask is that they be "reasonably" consistent. What this means is that each has an $\mathscr{A}$, constructed as in Theorem 1, which is reasonably consistent.

If we look at the $\overline{\mathscr{A}}_i$, $1 \leqslant i \leqslant 5$, and $\overline{\mathscr{E}}$ in the application in Section 4, all are "reasonably" consistent. In fact $\overline{\mathscr{A}}_2$, $\overline{\mathscr{A}}_3$, and $\overline{\mathscr{A}}_5$ are consistent. Let us look at $\overline{\mathscr{A}}_1$ to see how it is "reasonably" consistent. From Theorem 1, for $\overline{\mathscr{A}}_1$ to be consistent we need $[\beta_{ij}, \gamma_{ij}] \subset [\beta_{ik}, \gamma_{ik}] \cdot [\beta_{kj}, \gamma_{kj}]$, for all i, j, k. Consider $i = 1$, $k = 2$ and $j = 3$. We see that $[\beta_{13}, \gamma_{13}] = 1/2$, $[\beta_{12}, \gamma_{12}] = [1/5, 1/3]$, and $[\beta_{23}, \gamma_{23}] = 3$ and $1/2 \notin [3/5, 1]$. But since $1/2$ is "reasonably" close to $3/5$ we conclude $\overline{a}_{12} \cdot \overline{a}_{23}$ is "reasonably" close to $\overline{a}_{13}$. In $\overline{\mathscr{A}}_1$ we find that $\overline{a}_{ik} \cdot \overline{a}_{kj}$ is "reasonably" close to $\overline{a}_{ij}$ for all i, k, j and we conclude that $\overline{\mathscr{A}}_1$ is "reasonably" consistent. We have no test for reasonably consistent for fuzzy, positive, reciprocal matrices as is used for crisp, positive, reciprocal matrices.

4.2. Ranking fuzzy numbers

We end up (Eq. (33)) with fuzzy numbers $\overline{r}_1, \ldots, \overline{r}_m$ which need to be ranked so we may obtain the final ranking of the alternatives. Let H_1 be all the undominated fuzzy numbers $\overline{r}_i$. We say $\overline{r}_i$ is undominated if no $\overline{r}_j > \overline{r}_i$, $j \neq i$. Next define H_2 to be all the undominated $\overline{r}_k$ after deleting all the fuzzy numbers in H_1. Similarly, we construct $H_3, \ldots, H_d$. Then, all the A_i corresponding to a $\overline{r}_i$ in H_1 have the highest ranking, all the A_j having $\overline{r}_j$ in H_2 have the second ranking, etc. Properties of this ranking method are given in [12,20].

5. Results

We first found the fuzzy weights vectors $\overline{w}_k$ for A_k, $1 \leqslant k \leqslant 5$, using the formulas in Section 3.1. To obtain the fuzzy weight vector for $\overline{\mathscr{E}}$ we applied our EA (Method I of Section 3). All fuzzy numbers were calculated for α-cuts of $\alpha = 0, 0.2, 0.4, 0.6, 0.8, 1.0$. Instead of displaying all these data we show, in Table 3, only $\alpha = 0$ and $\alpha = 1$. The final fuzzy weights $\overline{r}_1, \overline{r}_2, \overline{r}_3$ are given in Fig. 4. We see that $H_2 = \{A_1, A_3\}$, $H_1 = \{A_2\}$ and the student selected A_2.

6. Summary and conclusions

We directly fuzzified Saaty's original procedure of computing the weights in HA to get the fuzzy weights in FHA. We checked our method against the 3×3, and 4×4 cases, where formulas exist for the weights, to show we are obtaining the correct fuzzy weights. Therefore, we believe we now have the correct FHA.

The calculation of the fuzzy weights (Eqs. (6)–(8)) is quite complicated so we used an EA (Appendix A) to search for the min (Eq. (7)) and the max (Eq. (8)). Future research will be concerned with obtaining a more efficient procedure of getting the fuzzy weights.

Appendix A

In this appendix we discuss the design of the evolutionary algorithm (EA) used for obtaining the fuzzy eigenvector $\overline{w}$ of a fuzzy, positive, reciprocal matrix $\overline{\mathscr{A}}$. Evolutionary algorithms perform a directed search. Therefore, they are useful for this kind of optimization problem, since there are no algorithms to compute the optimal values (Eqs. (7) and (8)).

Table 3
Fuzzy weights in the application

	$\overline{w}_{11}$	$\overline{w}_{12}$	$\overline{w}_{13}$		
For $\overline{\mathscr{A}}_1$					
$\alpha = 0$	[0.1158, 0.1630]	[0.5401, 0.6849]	[0.1990, 0.2967]		
$\alpha = 1$	[0.1219, 0.1549]	[0.5973, 0.6486]	[0.2296, 0.2478]		
	$\overline{w}_{21}$	$\overline{w}_{22}$	$\overline{w}_{23}$		
For $\overline{\mathscr{A}}_2$					
$\alpha = 0$	[0.1111, 0.1996]	[0.3776, 0.4754]	[0.3776, 0.4753]		
$\alpha = 1$	[0.1429, 0.1429]	[0.4286, 0.4286]	[0.4286, 0.4286]		
	$\overline{w}_{31}$	$\overline{w}_{32}$	$\overline{w}_{33}$		
For $\overline{\mathscr{A}}_3$					
$\alpha = 0$	[0.4443, 0.4999]	[0.4442, 0.4999]	[0.0476, 0.0666]		
$\alpha = 1$	[0.4509, 0.4706]	[0.4706, 0.4902]	[0.0556, 0.0625]		
	$\overline{w}_{41}$	$\overline{w}_{42}$	$\overline{w}_{43}$		
For $\overline{\mathscr{A}}_4$					
$\alpha = 0$	[0.2101, 0.4406]	[0.4723, 0.6945]	[0.0653, 0.1184]		
$\alpha = 1$	[0.2158, 0.2158]	[0.6818, 0.6818]	[0.1024, 0.1024]		
	$\overline{w}_{51}$	$\overline{w}_{52}$	$\overline{w}_{53}$		
For $\overline{\mathscr{A}}_5$					
$\alpha = 0$	[0.0834, 0.1255]	[0.4332, 0.5000]	[0.3977, 0.4666]		
$\alpha = 1$	[0.1002, 0.1111]	[0.4332, 0.4444]	[0.4444, 0.4660]		
	$\overline{e}_1$	$\overline{e}_2$	$\overline{e}_3$	$\overline{e}_4$	$\overline{e}_5$
For $\overline{\mathscr{E}}$					
$\alpha = 0$	[0.0720, 0.1158]	[0.0371, 0.0695]	[0.3183, 0.3670]	[0.0872, 0.1328]	[0.3816, 0.4276]
$\alpha = 1$	[0.0836, 0.1050]	[0.0430, 0.0525]	[0.3231, 0.3661]	[0.0994, 0.1215]	[0.3850, 0.4245]

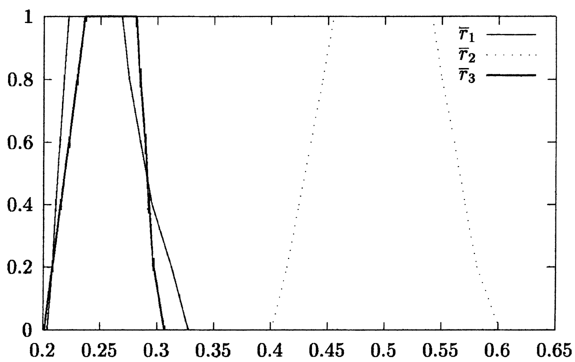

Fig. 4. The final fuzzy weights in the application.

The overall way of working of evolutionary algorithms is adopted from nature where the fittest members of a population have better chances to survive than other population members. The main operators of evolution are selection, which selects the fittest members of the population, and the recombination and mutation operators which are needed to create a new generation out of the fittest population members selected by the selection operator. These operators are then applied to the new generation. The process continues until a predefined number of generations is reached or some fitness limit is obtained.

Let us now discuss our EA in more detail. Since the aim is to find the fuzzy eigenvector $\overline{w}$ of an $m \times m$, positive, reciprocal, fuzzy matrix $\overline{\mathscr{A}}$, we have to run the EA for each k, $1 \leqslant k \leqslant m$, for each α-cut to obtain $w_{k1}(\alpha)$ and $w_{k2}(\alpha)$ (see Eqs. (7) and (8)).

For each EA we first create an initial population consisting of $P = 300$ elements. Each element of the population represents a crisp matrix A_p, $1 \leqslant p \leqslant P$. Since the A_p has to be a positive, reciprocal matrix only the upper triangular matrix without the diagonal ($a_{ii} = 1$, $1 \leqslant i \leqslant m$) has to be used for coding an element of the population. Therefore, A_p is randomly generated by

$$a_{ij} \in \overline{a}_{ij}(\alpha) \tag{A.1}$$

for $1 \leqslant i < j \leqslant m$. In order to obtain a positive, reciprocal matrix we simply have to set $a_{ii} = 1$, $1 \leqslant i \leqslant m$ and $a_{ji} = a_{ij}^{-1}$ for $1 \leqslant i < j \leqslant m$. Code each matrix A_p into

$$\pi_p = (a_1, a_2, a_3, \ldots, a_N, \sigma) \tag{A.2}$$

with $a_1 = a_{12}$, $a_2 = a_{13}$, $a_3 = a_{23}, \ldots, a_N = a_{m-1,m}$ ($N = 0.5 \cdot m \cdot (m-1)$) and σ stands for the mutation rate of the corresponding element, which is initially set to 0.3.

After initializing the evolution starts with the selection process. Here the kth component of the eigenvector of corresponding population member is computed as discussed in Section 2 (Eqs. (4) and (5)). These values stand for the fitness of the corresponding element, since $w_{k1}(\alpha)$ ($w_{k2}(\alpha)$) have to be optimized. For evaluating $w_{k1}(\alpha)$ those 45 members of the population are selected having the smallest values, and for $w_{k2}(\alpha)$ those 45 members are chosen having the largest values.

The recombination process now builds a temporary generation by applying a crossover operator to the previously selected individuals (of the previous generation). For each member of the temporary generation two "parents" $\pi_p^{\text{old}_1}$ and $\pi_p'^{\text{old}_2}$ are randomly chosen from the 45 fittest elements of the previous generation. In order to get many different individuals a generalized multipoint crossover operator is used. According to the user defined crossover probability, which in the experiments was set to 0.875 a number $q = \lceil 0.875N \rceil$ is generated. Now, q crossover points c_i ($1 \leqslant i \leqslant q$) are chosen randomly in $\{1, 2, \ldots, N\}$. After ordering the crossover points and initially set $c_0 = 0$ and $c_{q+1} = N$ we get $0 = c_0 < c_1 \leqslant c_2 \leqslant \cdots \leqslant c_q \leqslant c_{q+1} = N$. Now, another value $r \in [0, 1]$ is randomly chosen so that between two crossover points c_o and c_{o+1} the new matrix is build by using the equations:

$$a_n^{\text{temp}_1} = a_n^{\text{old}_1} + r * (a_n^{\text{old}_2} - a_n^{\text{old}_1}), \tag{A.3}$$

$$a_n^{\text{temp}_2} = a_n^{\text{old}_2} + r * (a_n^{\text{old}_1} - a_n^{\text{old}_2}) \tag{A.4}$$

for $0 < c_o < n \leqslant c_{o+1} \leqslant N$, $o = 1, \ldots, q$. The mutation rate σ is also adjusted during the recombination process by using the same operator and we get

$$\sigma^{\text{temp}_1} = \sigma^{\text{old}_1} + r * (\sigma^{\text{old}_2} - \sigma^{\text{old}_1}), \tag{A.5}$$

$$\sigma^{\text{temp}_2} = \sigma^{\text{old}_2} + r * (\sigma^{\text{old}_1} - \sigma^{\text{old}_2}) \tag{A.6}$$

for a randomly chosen $r \in [0, 1]$. Now, one of these two elements is randomly chosen (called π_p^{temp}) and put into the temporary generation. By applying Eqs. (A.3) and (A.5) 300-times to randomly chosen parents we get the whole temporary generation.

The mutation process finally generates the new generation by randomly changing the members $\pi_p^{\text{temp}} = (a_1^{\text{temp}}, \ldots, a_N^{\text{temp}}, \sigma^{\text{temp}})$ of the temporary generation. One by one each individual is modified according to its own mutation rate. The mutation rate σ^{temp} of each member is first modified. Here we used the equation

$$\sigma^{\text{new}} = \sigma^{\text{temp}} + \exp(\tau * N(0,1)), \tag{A.7}$$

where $N(0,1)$ stands for a normally distributed random variable having expectation value 0 and standard deviation 1. τ is an additional parameter which is set to $n^{-0.5}$ [1]. The global factor $\exp(\tau N(0,1))$ allows for an overall change of the mutability for each individual. Each element a_n $(1 \leqslant n \leqslant N)$ of the individual is mutated by computing

$$a_n^{\text{new}} = a_n^{\text{temp}} + \sigma^{\text{new}} * N(0,1) \tag{A.8}$$

for $1 \leqslant n \leqslant N$. Here we also have to check whether a_n^{new} lies in the α-cut of the corresponding $\overline{a}_{ij}$ or not. If so, mutation continues otherwise Eq. (A.8) is applied to a_n^{temp} until $a_n^{\text{temp}} \in \overline{a}_{ij}[\alpha]$ holds. By repeating this process for each temporary individual we get the entire new generation π_p.

Now again the selection operator finds the best members of the population, which are used by the recombination operator to produce a temporary generation and the mutation operator which mutates each temporary element in order to construct the next generation. As we have already mentioned this process continues until a given number of generations is reached, or a given fitness level is obtained.

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